**LINEAR FUNCTIONS**

A linear function is a set of points that when joined form a straight line. Each point has two components: an x-value and a y-value usually written in the form (x, y).

If we know two points located on a coordinate plane we can determine a specific amount of information. This includes:

**Slope** is represented by the variable m and has formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Slope is defined as the steepness of the line. Lines that move up from left to right have a positive slope, lines that move down from left to right have a negative slope, vertical lines have no slope and horizontal lines have zero slopes.

**Midpoint** is represented by the variable M and has formula: $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Midpoint is the point located half way between two given points. It may be referred to as the location of a bisector.

**Distance** is represented by the variable d and has formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Distance is the how far the two points are apart. This answer is always positive.

**Slope of parallel lines -** lines that are parallel have exactly the same slope. This is represented as $m_1 = m_2$

**Slope of perpendicular lines -** lines that are perpendicular have slopes that are negative reciprocals of one another. This is represented by $m_1 \cdot m_2 = -1$

For example: Given the points (3, 7) and (-9, -3) find:

a) slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{7-(-3)}{3-(-9)}$$
$$m = \frac{7+3}{3+9}$$
$$m = \frac{10}{12} = \frac{5}{6}$$

b) midpoint

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$M\left(\frac{3+(-9)}{2}, \frac{7+(-3)}{2}\right)$$
$$M\left(\frac{-6}{2}, \frac{4}{2}\right)$$
$$M(-3,2)$$

c) distance

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(3-(-9))^2 + (7-(-3))^2}$$
$$d = \sqrt{(3+9)^2 + (7+3)^2}$$
$$d = \sqrt{(12)^2 + (10)^2}$$
$$d = \sqrt{144+100}$$
$$d = \sqrt{244} = 2\sqrt{61} = 16.97$$

d) slope of parallel and perpendicular lines

$$m_1 = \frac{5}{6}$$
$$parallel \quad m_1 = m_2$$
$$m_2 = \frac{5}{6}$$
$$perpendicular \quad m_1 m_2 = -1$$
$$m_2 = -\frac{6}{5}$$

To determine a set of points we must have some rule that regulates or governs the relationship between the points. The rule is called an **equation** and is given in the form **Ax + By = C**. In assignments and in texts the equation is usually written in the form of 3x - 6y = 12 or -x + y = 2 or

5x - y = 11 etc. Therefore we can conclude that:

**a)** if we know two points we can do the above calculations

**b)** if we know an equation we can find two points so that we can do the above calculations.

For example: Given 2x + 3y = 6

a) if we choose to let x = -3 then

**2(-3) + 3y = 6**

**-6 + 3y = 6** The calculated point is (-3, 4)

**3y = 12**

**y = 4**

b) similarly, if we choose to let x = 15, then y = -8 and the point would be (15, -8)

Once two points have been calculated we can determine:

a) slope b) midpoint c) distance d) slope of parallel and perpendicular lines

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{4-(-8)}{-3-15}$$
$$m = \frac{4+12}{-18}$$
$$m = \frac{16}{18} = \frac{8}{9}$$

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$M\left(\frac{(-3)+15}{2}, \frac{4+(-8)}{2}\right)$$
$$M\left(\frac{12}{2}, \frac{-4}{2}\right) = M(6,-2)$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(-3)-15)^2 + (4-(-8))^2}$$
$$d = \sqrt{(-18)^2 + (4+12)^2}$$
$$d = \sqrt{(-18)^2 + (16)^2}$$
$$d = \sqrt{324+256}$$
$$d = \sqrt{580} = 2\sqrt{145} = 24.08$$

$$m_1 = \frac{8}{9}$$
$$parallel \quad m_1 = m_2$$
$$m_2 = \frac{8}{9}$$
$$perpendicular \quad m_1 m_2 = -1$$
$$m_2 = -\frac{9}{8}$$

Two special locations on a graph are the points where the line cuts the x and y axis. The point where the line cuts the y-axis is referred to as the **y-intercept**. At this point the value of x is always zero and y can be any value. This point can be labelled (0, y) but the usual standard is (0, b). The point where the line cuts the x-axis is referred to as the **x-intercept** and at this point y is always equal to zero and x can be any value. This point is usually labelled (x, 0).

To find the x-intercept from an equation we let y = 0 and solve for one positive x.

Example: 5x - 7y = 15 **5x - 7(0) = 15**

**5x = 15** (3, 0)

**x = 3**

To find the y-intercept from an equation we have two different procedures:

a) let x = 0 and solve for one positive y.

Example: 3x - 4y = 12 **3(0) – 4y = 12**

**-4y = 12** (0, -3)

**y = -3**

b) the second way is to solve the equation for one positive y. This will put the equation in the form **y = mx + b**.(slope – intercept formula). Doing this achieves to results:

i) we can identify b which is the y-intercept and can be written as b = __ or as (0, b)

ii) we can identify "m" which is the slope of the equation. This represents a second procedure for determining the slope of a line. As we did above we could find two points and then calculate slope or we can solve for y and use the coefficient (number in front) of x as our slope.

Example: 5x + 2y = 8

$$2y = -5x + 8$$
$$y = \frac{-5}{2}x + \frac{8}{2}$$
$$y = -\frac{5}{2}x + 4$$

⟹ $m = -\frac{5}{2}, b = 4, (0,4)$

Doing all the above represents ways in which we determine specific information about a line but we must be able to determine the equation of the line.

What happens if we know specific information (slope and y-intercepts, slope and points, just points, points and relationships, or equations and relationships) and we are asked to determine the equation that would give us this information. The key to all the following exercises is the need to know **a point** that lies on the line and the **slope of the line**. This means that anytime you are given any information sufficient information must be present so that you can determine the slope and a point on the line. It is necessary to write all equations in the form **Ax + By = C** which in some cases will require isolating the constant term and the removal of any fractional coefficients or constants.

**The Slope Intercept Formula** y = mx + b

This formula is used when we know:

a) the slope (m) and the y-intercept (b)

b) the slope (m) and the point containing the y-intercept (0, b)

Category 1: we know slope and the y-intercept

Given: m = 5 and y-intercept is 6 (b = 6)

$$y = mx + b$$
$$y = 5x + 6$$
$$-5x + y = 6$$

Category 2: we know the slope and the point containing the y-intercept

Given: m= 5/7 and the point (0, -2). Critical idea is to recognize that the point contains "b"

$$y = mx + b$$
$$y = \frac{5}{7}x + (-2)$$
$$7y = 7 \cdot \frac{5}{7}x + (-2) \cdot 7$$
$$7y = 5x - 14$$
$$-5x + 7y = -14$$

**The Point Slope Formula** $(y_2 - y_1) = m(x_2 - x_1)$

Remember the keys are knowing the slope (m) and what point (x, y) I want to use.

Note: 1. substitute the values of "x" and "y" from the given or calculated point into the variables identified with subscript "I"

2. replace the variables with subscript "2" with an "x" and "y"

3. if the slope is a fraction multiply both sides by the denominator prior to removing the parenthesis.

4. if the substituted value for "x" or "y" is a fraction, first remove the parenthesis and then multiply each term by a common denominator.

Category 3: where we know slope and the point containing the y-intercept

Given: m = 5 and the point (0, 6)

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y - 6) = 5(x - 0)$$
$$y - 6 = 5x$$
$$-5x + y = 6$$

Category 4: where we know the slope and a point

Given: m = 2/3 and the point is (2, 3)

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y - 3) = \frac{2}{3}(x - 2)$$
$$3 \cdot (y - 3) = 3 \cdot \frac{2}{3}(x - 2)$$
$$3y - 9 = 2x - 4$$
$$-2x + 3y = 5$$

Category 5: where we know two points

Given: 2 points (3, 7) and (-1, -9)

a) determine the slope (remember that this is a requirement for writing an equation)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{7 - (-9)}{3 - (-1)}$$
$$m = \frac{7 + 9}{3 + 1}$$
$$m = \frac{16}{4} = 4$$

b) use the slope from part (a) and choose one of the points. Remember that it does not matter which does not matter which point you choose, either one will give you the correct answer.

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y - 7) = 4(x - 3)$$
$$y - 7 = 4x - 28$$
$$-4x + y = -21$$

Category 6: knowing a point and given the equation of a line parallel to or perpendicular to the line through the known point.

Given: point (3, 5) and the equation 2x + y = 6

a) First solve the equation for one positive y (form y = mx + b)

$$2x+y=6$$
$$y=-2x+6$$

if question calls for parallel lines use m as: **m = -2**

if question calls for perpendicular lines use m as: **m = 1/2**

b) use the appropriate slope and the known point and use the point slope formula

1. for parallel lines

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y-5) = -2(x-3)$$
$$y - 5 = -2x + 6$$
$$y = -2x + 11$$

2. for perpendicular lines

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y-5) = \frac{1}{2}(x-3)$$
$$2 \cdot (y-5) = 2 \cdot \frac{1}{2}(x-3)$$
$$2y - 10 = x - 3$$
$$2y = x + 7$$

Category 7: knowing a point and given two other points which define a line that is parallel to or perpendicular to the line through the known point.

Given: point (2, 6) and the other line through (1,4) and (5, 10)

a) First determine the slope of the line joining the two points using the slope formula

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$m = \frac{(4-10)}{(1-5)}$$
$$m = \frac{-6}{-4} = \frac{3}{2}$$

if question calls for parallel lines use m as: m = 3/2

if question calls for perpendicular lines use m as: m = -2/3

b) use the appropriate slope and the known point and use the point slope formula

1. for parallel lines

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y-6) = \frac{3}{2}(x-2)$$
$$2 \cdot (y-6) = 2 \cdot \frac{3}{2}(x-2)$$
$$2y - 12 = 3x - 6$$
$$2y = 3x + 12$$

2. for perpendicular lines

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y-6) = -\frac{2}{3}(x-2)$$
$$3 \cdot (y-6) = 3 \cdot \frac{-2}{3}(x-2)$$
$$3y - 18 = -2x + 4$$
$$3y = -2x + 22$$

Category 8 - perpendicular bisector of a line segment defined by two points

Given: line segment defined by the points (-2, 7) and (9, -4)

a) First determine the slope of the line segment joining the two points using the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-(-4)}{-2-9} = \frac{7+4}{-11} = \frac{11}{-11} = -1$$

b) Next determine the slope of perpendicular lines using the definition of perpendicular lines $m_1 * m_2 = -1$

$$m_1 \cdot m_2 = -1$$
$$-1 \cdot m_2 = -1$$
$$m_2 = 1$$

c) Determine the midpoint of the line segment using the midpoint formula

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-2+9}{2}, \frac{7+(-4)}{2}\right) = \left(\frac{-7}{2}, \frac{3}{2}\right)$$

d) Determine the equation using the info from *b)* and *c)* and the point slope formula

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$\left(y - \frac{3}{2}\right) = 1\left(x - \left(\frac{-7}{2}\right)\right)$$
$$y - \frac{3}{2} = x + \frac{7}{2}$$
$$2 \cdot y - 2 \cdot \frac{3}{2} = 2 \cdot x + 2\frac{7}{2}$$
$$2y - 3 = 2x + 7$$
$$2y = 2x + 10$$

Category 9 - Special cases -- lines parallel to or perpendicular to the x or y axis through a particular point. Those coordinates serve as answer to the equation of the line.

Note: if a line is parallel to the y- axis it is a vertical line therefore the equation reads *x = some number*

if a line is parallel to the x-axis it is a horizontal line therefore the equation reads *y = some number*

if a line is perpendicular to the y- axis it is a horizontal line therefore the equation reads *y = some number*

if a line is perpendicular to the x-axis it is a vertical line therefore the equation reads *x = some number*

Examples: Determine the equation given the following:

1) through (-4, 5) and parallel to x-axis; **y = 5**
2) through (-4, 5) and perpendicular to x-axis **x = -4**
3) through (-2, -6) and parallel to y-axis **x = -2**