Limits

The Tangent Line Problem:

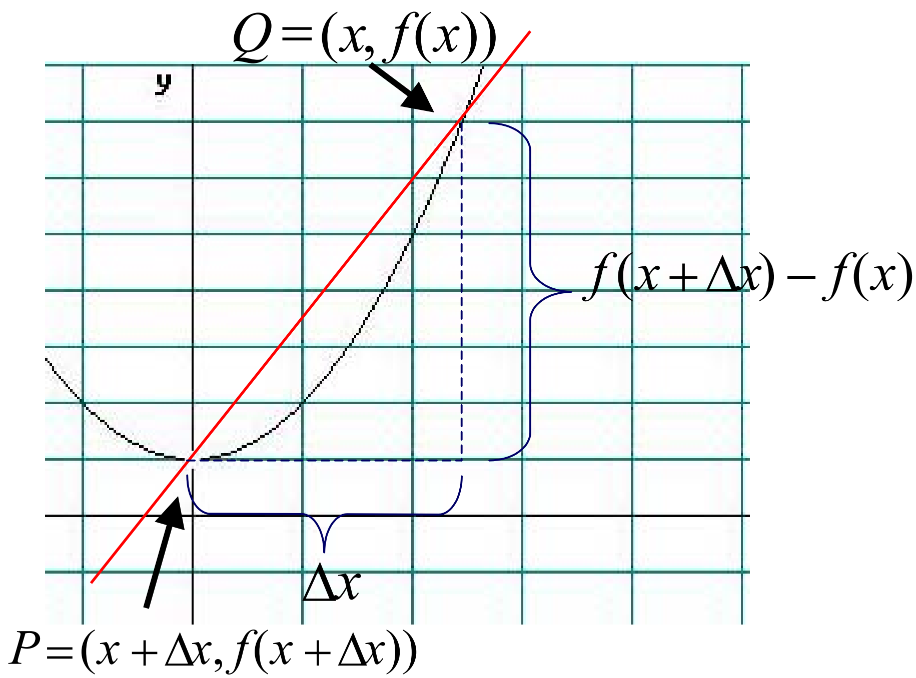
The problem of finding the tangent line at any point P is equivalent to finding the slope of the slope of the tangent line at P. Remember to write an equation of any line you must know the slope and a point on the line (it could be the y-intercept). The formulas that you would be using include:

a) the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$ {requires 2 points}

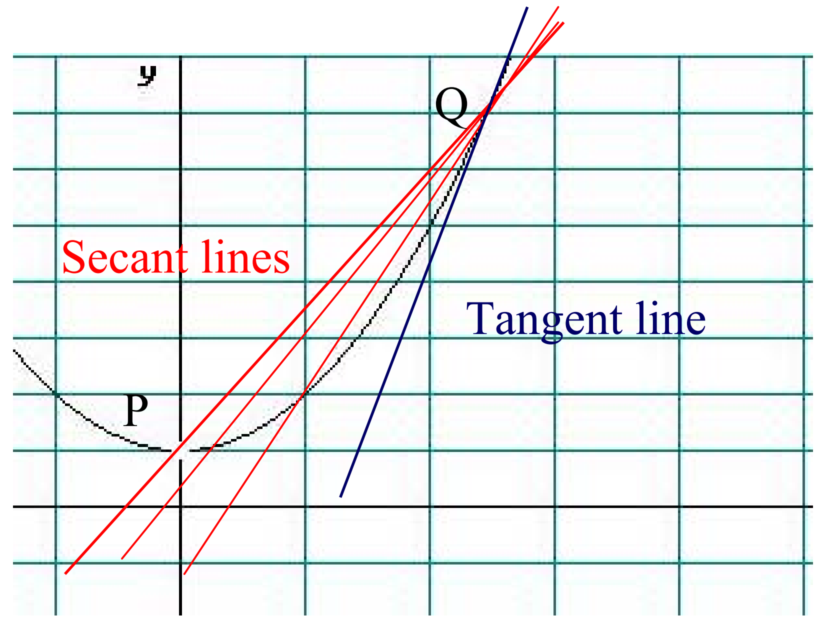
b) the point slope formula: $(y_2 - y_1) = m(x_2 - x_1)$

Since you will be analyzing the idea of the equation of a tangent line you will be given the point on a curve defined as (x, f(x))--the point of tangency, and you will have to determine a second point on the curve defined as (x + Δx, f(x + Δx)). The presence of the second point defines the existence of a secant line (line that cuts the curve in more than one place). Therefore, in very basic terms we will be using the slope of the secant line to get the best possible approximation for the slope of the tangent line. Utilizing the slope formula and the two identified points the slope of the tangent line will read as:

$$m_{\text{sec}} = \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



The secant line through PQ

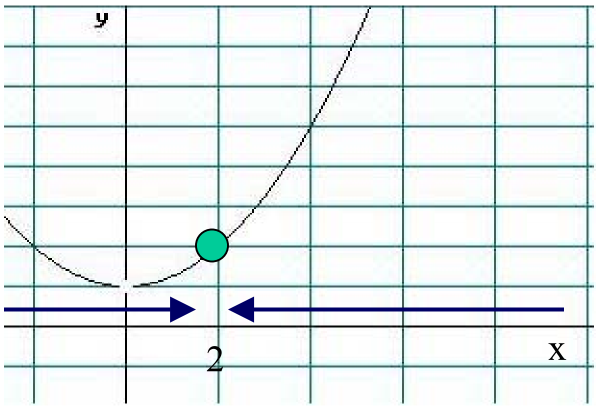


As P approaches Q, the secant lines approach the tangent line.

As the point P approaches the point Q, the slope of the secant line will approach the slope of the tangent line. When such a situation occurs we indicate that a "limiting position" exists and the slope of the tangent line is said to be the limit of the slope of the secant line

Informal Definition of a Limit:

Given the function $f(x) = \frac{1}{2}x^2 + 2$ we can examine the behavior of the graph of "f "as "x " gets near "2 " by creating two sets of x-values - one set that approaches x = 2 from the left and a second that approaches from the right



x	1.9	1.99	1.999	1.9999	2
f(x)	3.805	3.980	3.998	3.9998	?

x	2	2.0001	2.001	2.01	2.1
f(x)	?	4.0002	4.002	4.02	2.2

As you can see from the tables as "x " approaches 2 from the left, the value of the function f(x) approaches 4 and as "x " approaches from 2 from the right, the value of the function f(x) approaches 4

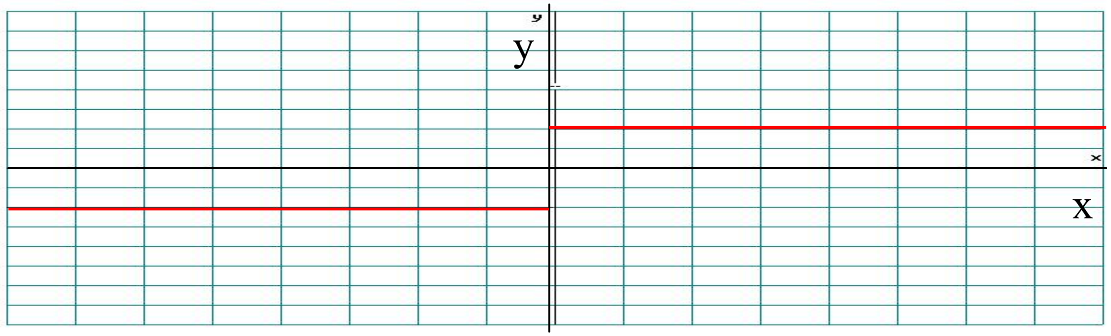
If f(x) becomes arbitrarily close to a single number "L" as x approaches c (some number being considered in the function) from either side (+ or -), then the limit of f(x) , as x approaches c, is "L".

$$\lim_{x \to c} f(x) = L$$

Limits That Fail to Exist:

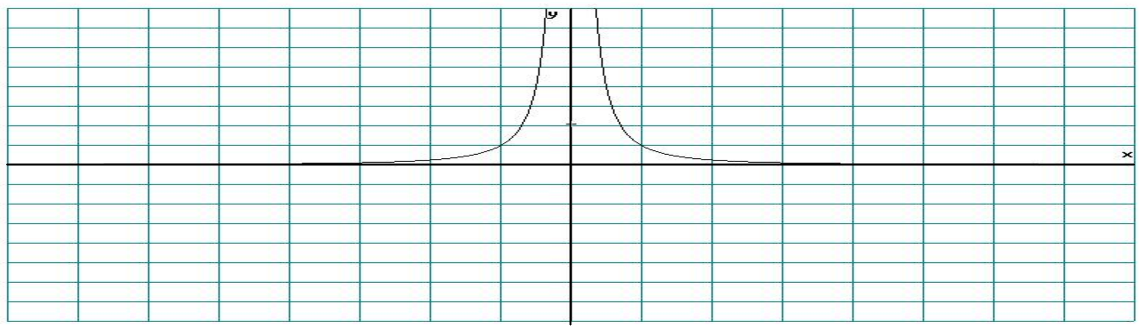
a) When the behavior from the right (+) differs from behavior from the left (-).

$$\lim_{x \to 0} \frac{|x|}{x}$$



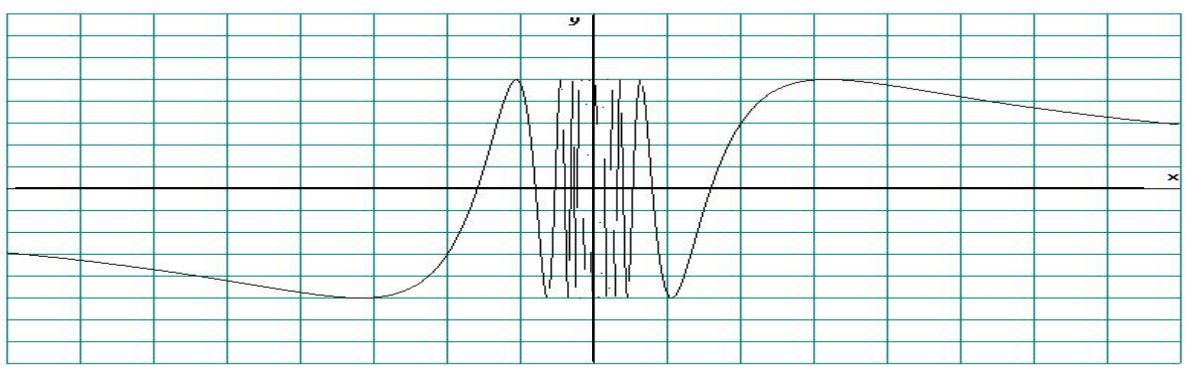
b) The function f(x) increases or decreases without bound (number gets larger and larger or smaller and smaller) and never reaches any real number "L".

$$\lim_{x \to 0} \frac{1}{x^2}$$



c) The function f(x) oscillates between two fixed values as x approaches c.

$$\lim_{x \to 0} \sin \frac{1}{x}$$



The Formal Definition of a Limit: (The ε–δ definition of a limit)

The definition is derived from an analysis of the informal definition and examining the two phrases:

a) "f(x) becomes arbitrarily close to L"

b) "x approaches c"

To give an exact definition to part (a) we select ε (epsilon) to represent a (small) positive number and give the expression the meaning that f(x) lies in the interval $(L - \varepsilon, L + \varepsilon)$. This can be written $|f(x) - L| < \varepsilon$, using absolute value.

The second phrase (b) means that there exists a positive number δ (delta) such that x lies in either the interval $(c - \delta, c)$ or in the interval $(c, c + \delta)$. This can be expressed as: $0 < |x - c| < \delta$.

To summarize the above points:

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement: $\lim_{x \to c} f(x) = L$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Using the Formal Definition

1. Finding a δ for a given ε

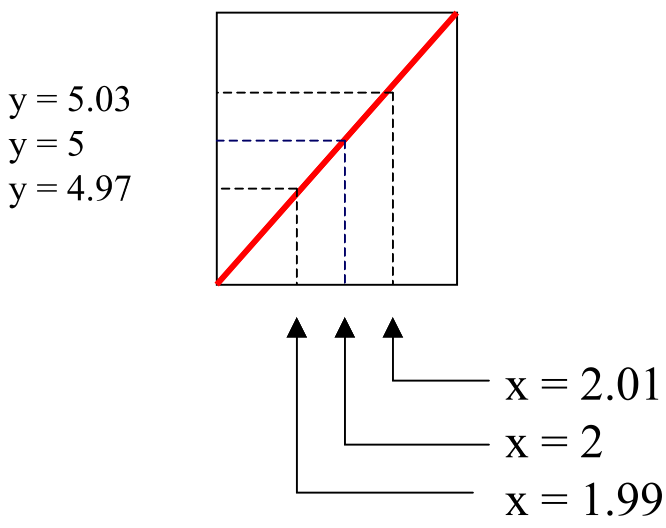
Given the limit $\lim_{x \to 2}(3x - 1) = 5$

find δ such that $|(3x - 1) - 5| < 0.03$ whenever $0 < |x - 2| < \delta$

Solution:

1. From the formal definition we can identify that $\varepsilon = 0.03$
2. The expression $|(3x - 1) - 5| = |3x - 1 - 5| = |3x - 6| = 3|x - 2|$
3. Since the inequality $|(3x - 1) - 5| < 0.03$ is equivalent to $3|x - 2| < 0.03$ you can choose $\delta = (1/3)(0.03) = 0.01$
4. A enlarged portion of the graph $f(x) = 3x - 1$ at $x = 2$ is provided below

In conclusion: Since $0 < |x - 2| < 0.01$ it implies that $|(3x - 1) - 5| = 3|x - 2| < 3(0.01) = 0.03$



Although we could find δ, this does not prove the existence of the limit. The next example demonstrates the existence of a limit -- you must prove you can a δ for any ε.

2. Proving the existence of a limit

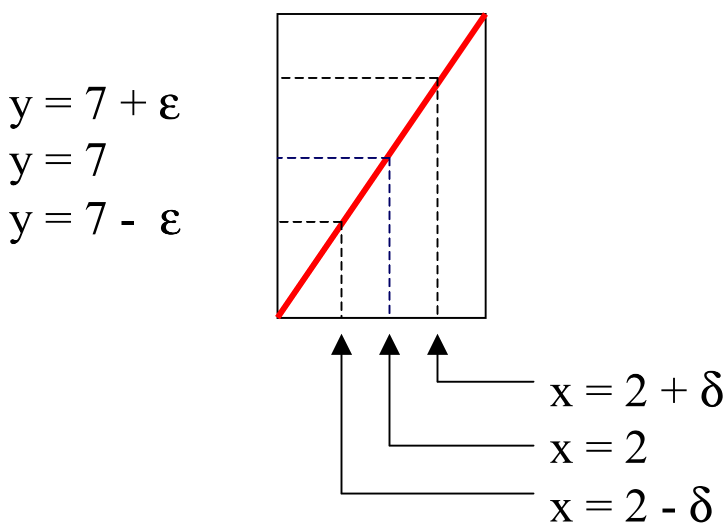
$$\lim_{x \to 2}(4x - 1) = 7$$

It must be shown that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(4x - 1) - 7| < \varepsilon$ whenever $0 < |x - 2| < \delta$. You choice for δ depends on ε, you need to establish a connection between the absolute values $|(4x - 1) - 7|$ and $|x - 2|$.

$$|(4x - 1) - 7| = |4x - 8| = 4|x - 2|$$

Thus $4|x - 2| < \delta$ and since $|x - 2| < \varepsilon$ it can be determined that $\delta = \varepsilon/4$ and this choice works because $0 < |x - 2| < \delta = \varepsilon/4$ and this implies

$$|(4x - 1) - 7| = 4|x - 2| < 4(\varepsilon/4) = \varepsilon$$



An enlarged portion of the graph of $f(x) = 4x-1$ at $x = 2$ is shown to the left

δε

Limits of Algebraic Functions

It has been shown that the limit of f(x) as x approaches c does not depend on the value of the f at x = c. There are situations when the limit is precisely f(c) and we can say that the limit is evaluated by the process of direct substitution and these functions can be described as continuous at "c "(concept explained in detail a little later)

$$\lim_{x \to c} f(x) = f(c)$$

Direct Substitution

Basic Limits

Let b and c be real numbers and let n be a positive integer

$$1.\lim_{x \to c} b = b \qquad 2.\lim_{x \to c} x = c \qquad 3.\lim_{x \to c} x^n = c^n$$

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiplication : $\lim_{x \to c} [bf(x)] = bL$
2. Sum or Difference : $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product : $\lim_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $k \neq 0$

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Examples:

1. $\lim_{x \to 3} 7 = 7$

2. $\lim_{x \to 5} x = 5$

3. $\lim_{x \to 4} x^2 = (4)^2 = 16$

4. $\lim_{x \to 3} (2x^2 - 5) = \lim_{x \to 3} 2x^2 - \lim_{x \to 3} 5$

$$= 2\left[\lim_{x \to 3} x^2\right] - \lim_{x \to 3} 5$$

$$= 2(3)^2 - 5$$

$$= 13$$

5. $\lim_{x \to 3} (3x+5)(4x-1) = \lim_{x \to 3} (3x+5) \lim_{x \to 3} (4x-1)$

$$= [(3(3)+5][4(3)-1]$$

$$= [14][11]$$

$$= 154$$

6. $\lim_{x \to 2} \frac{(3x+1)^3}{(5x-3)} = \frac{\left(\lim_{x \to 2} (3x+1)\right)^3}{\lim_{x \to 2} (5x-3)}$

$$= \frac{(3(2)+1)^3}{(5(2)-3)}$$

$$= \frac{7^3}{7} = 7^2 = 49$$

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and a is a real number, then

$$\lim_{x \to a} p(x) = p(c)$$

2. If r is a rational function given by $r(x) = p(x)/q(x)$ and a is a real number such that $q(a) \neq 0$, then

$$\lim_{x \to a} r(x) = r(a) = \frac{p(a)}{q(a)}$$

Examples:

1. $\lim_{x \to 2} 3x^2 - 5x + 2 = 3(2)^2 - 5(2) + 2 = 12 - 10 + 2 = 4$

Direct substitution is valid for polynomial functions

2. $\lim_{x \to 2} \frac{2x^2 + x - 3}{x + 2} = \frac{2(2)^2 + 2 - 3}{2 + 2} = \frac{8 + 2 - 3}{4} = \frac{7}{4}$

Direct substitution is valid for rational functions in situations where the denominator is not equal to zero

Limit of a function involving a radical

Let n be a positive integer. The following limit is valid for all a if n is odd, and is valid for a > 0 if n is even.

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

Examples:

1. $\lim_{x \to 32} \sqrt[5]{x} = \sqrt[5]{32} = 2$

2. $\lim_{x \to 8} \sqrt[4]{2x} = \sqrt[4]{2(8)} = \sqrt[4]{16} = 2$

Limit of a composite function

If f and g are functions such that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to a} f(g(x)) = f(L)$$

Example:

$$\lim_{x \to 3} \sqrt[3]{4x^2 - 9}$$

Solution: Let $g(x) = 4x^2 - 9$ and $f(x) = \sqrt[3]{x}$ because

$\lim_{x \to 3} g(x) = 27$ and $\lim_{x \to 27} f(x) = 3$

it follows that:

$$\lim_{x \to 3} \sqrt[3]{4x^2 - 9} = \lim_{x \to 3} f(g(x)) = f(27) = \sqrt[3]{27} = 3$$

Strategy for finding limits:

1. Learn to recognize which limits can be evaluated by using direct substitution
2. If the limit of f(x) as x approaches a cannot be evaluated by using direct substitution, try to find a function g that agrees with f for all other than x = a. In simple terms it means trying to find a g so that the limit of g(x) can be evaluated by direct substitution.
3. Utilize the following: "Functions that Agree at all but one Point"

   Let a be a real number and let f(x) = g(x) for all $x \neq c$ in an open interval containing a. If the limit of g(x) as x approaches a exists, then the limit of f(x) also exists and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$

4. Use a graph or table to reinforce your conclusion.

Example of two functions agreeing at all but one point

Given:

$$f(x) = \frac{x^3 + 4x^2 + 5x}{x}$$

$$g(x) = x^2 + 4x + 5$$

Show that these two functions have the same values for all x other than x = 0

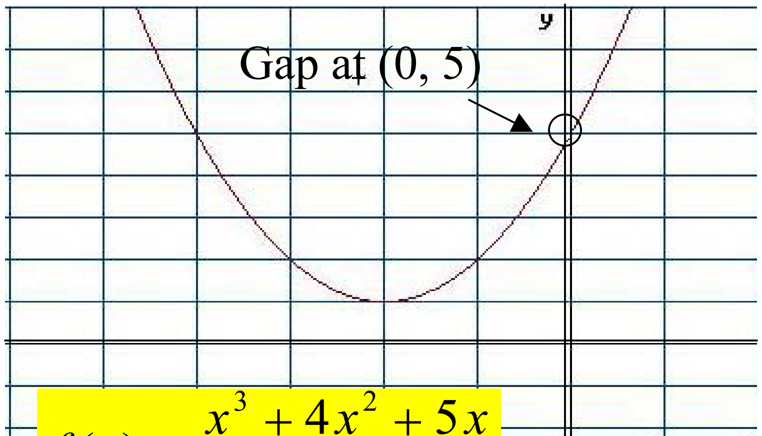
Factoring the numerator of f(x)

$$f(x) = \frac{x^3 + 4x^2 + 5x}{x} = \frac{x(x^2 + 4x + 5)}{x}$$

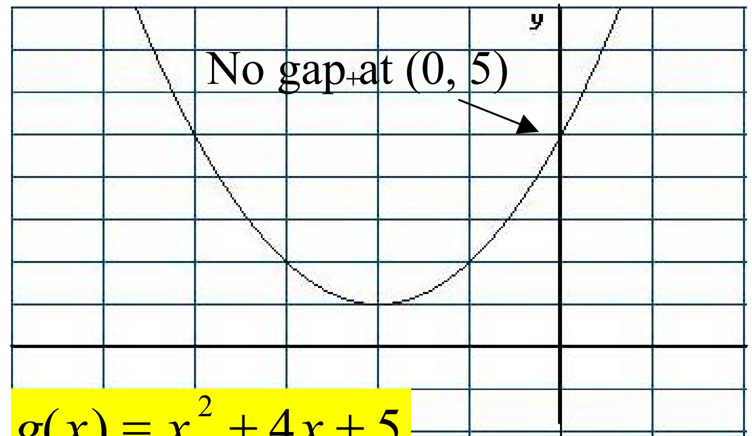
If $x \neq 0$, you can cancel like factors to obtain

$$f(x) = \frac{\cancel{x}(x^2 + 4x + 5)}{\cancel{x}} = x^2 + 4x + 5 = g(x), x \neq 0$$

(See graphs of functions on next slide)



$f(x) = \dfrac{x^3 + 4x^2 + 5x}{x}$



$g(x) = x^2 + 4x + 5$

These two graphs are identical for all points except x = 0.

Using the arguments of this example we can find the limit of f(x)by applying Functions that Agree at all but one Point"

$$\lim_{x \to 0} \frac{x^3 + 4x^2 + 5x}{x} = \lim_{x \to 0} \frac{x(x^2 + 4x + 5)}{x}$$ (Factor)

$$= \lim_{x \to 0} \frac{x(x^2 + 4x + 5)}{x}$$ (Cancel like factors)

$$= \lim_{x \to 0} (x^2 + 4x + 5)$$ (Apply statement Functions that Agree at all but one Point")

$$= ((0)^2 + 4(0) + 5)$$ (Direct Substitution)

$$= 5$$ (Simplify)

Cancellation and Rationalization Techniques

In certain cases direct substitution creates a meaningless fractional form 0/0. Such an expression is called an indeterminate form because you cannot determine the limit from the form alone. When you encounter this form you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to rewrite the fraction is by canceling like factors and the second is by rationalizing the numerator.

Of note: The Factor Theorem states that if a is a zero of a polynomial function, then (x - a) is a factor of the polynomial. Therefore, if direct substitution is applied to a rational function and the result is the fractional form 0/0 it can be concluded that the factor (x - a) must be common to both the numerator and denominator of the rational expression.

Examples of cancellation

1. $\lim_{x \to 3} \frac{x^2 - 7x + 12}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x-4)}{(x-3)} = \lim_{x \to 3}(x-4) = -1$

2. $\lim_{x \to -2} \frac{x^3 + 8}{x^4 - 16} = \lim_{x \to -2} \frac{(x+2)(x^2 - 2x + 4)}{(x+2)(x-2)(x^2+4)} = \lim_{x \to -2} \frac{(x^2 - 2x + 4)}{(x-2)(x^2+4)} = \frac{12}{-32} = -\frac{3}{8}$

3. $\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25} = \lim_{x \to 25} \frac{(\sqrt{x} - 5)}{(\sqrt{x} - 5)(\sqrt{x} + 5)} = \lim_{x \to 25} \frac{1}{(\sqrt{x} + 5)} = \frac{1}{\sqrt{25} + 5} = \frac{1}{10}$

Examples of Rationalization

1. $$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$$

$$=\lim_{x\to 0}\frac{(\sqrt{x+1}-1)}{x}\cdot\frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$$

$$=\lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)}$$

$$=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}$$

$$=\frac{1}{\sqrt{x+1}+1}=\frac{1}{2}$$

2. $$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}$$

$$=\lim_{x\to 0}\frac{(\sqrt{3+x}-\sqrt{3})}{x}\cdot\frac{(\sqrt{3+x}+\sqrt{3})}{(\sqrt{3+x}+\sqrt{3})}$$

$$=\lim_{x\to 0}\frac{3+x-3}{x(\sqrt{3+x}+\sqrt{3})}$$

$$=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{3+x}+\sqrt{3})}$$

$$=\frac{1}{\sqrt{3+x}+\sqrt{3}}=\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}$$

3. $$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{(\sqrt{x+1}-2)}{(x-3)}\frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)}=\lim_{x\to 3}\frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}=$$

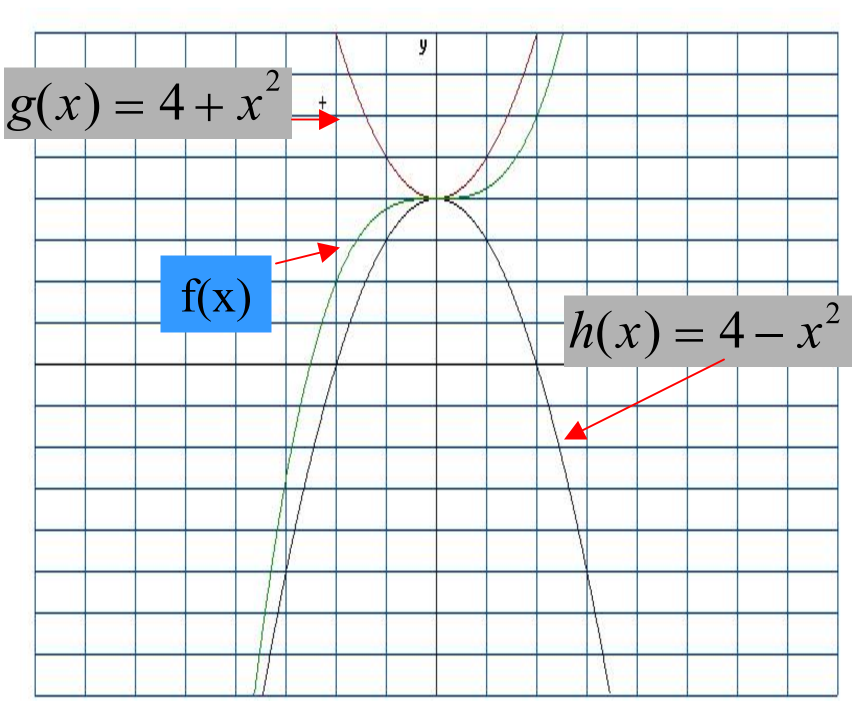
$$\lim_{x\to 3}\frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+1}+2)}=\lim_{x\to 3}\frac{1}{(\sqrt{x+1}+2)}=\frac{1}{4}$$

The Squeeze Theorem

This theorem focuses on the limit of a function that is squeezed (sandwiched) between two other functions, each of which has the same limit at a given x-value.

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$ then $\lim_{x\to c} f(x)$ exists and is equal to L.

Example: Find $\lim_{x\to c} f(x)$ when $c = 0$ and $(4 - x^2) \le f(x) \le (4 + x^2)$



$$\lim_{x\to 0} h(x) = \lim_{x\to 0}(4 - x^2) = 4$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(4 + x^2) = 4$$

$$\therefore \lim_{x\to c} f(x) = \lim_{x\to 0} f(x) = 4$$

because $h(x) \le f(x) \le g(x) \Rightarrow 4 \le f(x) \le 4$

Continuity

To say that a function is continuous at $x = c$ means that there is no interruptions in the graph of "f" at "c". That is, the graph is unbroken at "c" and there are no holes or gaps.

Definition of Continuity:

a) Continuity at a point: A function is continuous at "c" if the following three conditions are met:

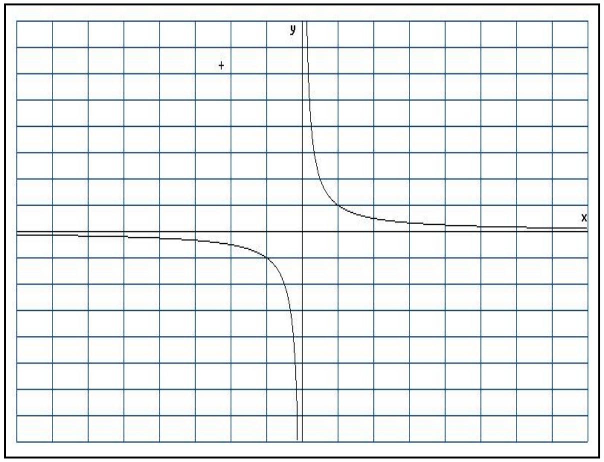
1) f(c) is defined
2) $\lim_{x \to c} f(x)$ exists
3) $\lim_{x \to c} f(x) = f(c)$.

b) Continuity on an Open Interval: A function is continuous on an open interval (a, b) if it is continuous at each point on the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is everywhere continuous.

If a function is defined on an interval "I" (except possibly at "c") and "f" is not continuous at "c", then "f" is said to have discontinuity at "c". Discontinuities fall into two categories: removable and non-removable. A discontinuity at "c" is called removable if "f" can be made continuously appropriately defining or redefining f(c) .

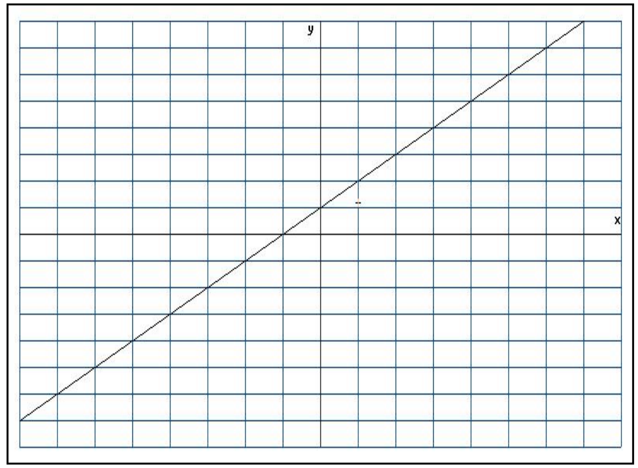
Examples of Continuity

$$f(x) = \frac{1}{x}$$



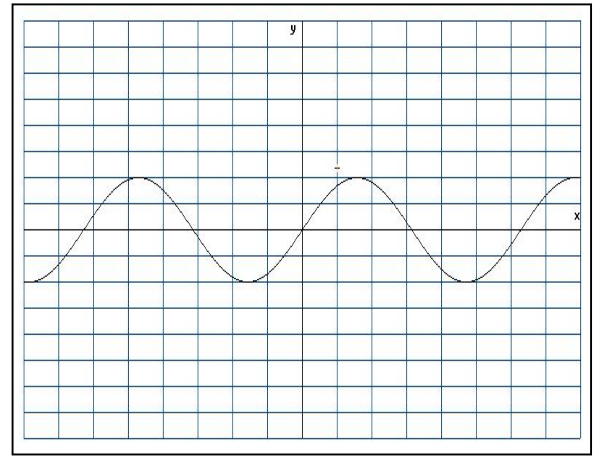
The domain of f is all nonzero real numbers.The function is continuous at every x in its domain. At x = 0 the functions has a non-removable discontinuity. In simple terms there is no way to make this function continuous at x = 0.

$$f(x) = \frac{x^2 - 1}{x - 1}$$



The domain of f is all real numbers except x = 1. By evaluating the limit as x goes to one and canceling factors we can conclude that f is continuous at every x value on its domain. At x = 1 the function has a removable discontinuity. The newly determined function is defined at f(1) = 2 and is continuous for all real numbers.

$$f(x) = \sin x$$



The domain of the function f is all real numbers The function is continuous on its entire domain.

Properties of Continuity

If “b” is a real number and “f ” and “g ” are continuous at $x = c$, then the following functions are also continuous at “c”.

1. Scalar multiplication b f
2. Sum and difference $f \pm g$
3. Product f g
4. Quotient f/g, if $g(c) \neq 0$

The following functions are continuous at every point in their domain:

1. Polynomial functions: $f(x) = ax^n + a_2x^{n-1} + \ldots + a_{n-1}x + a_n$
2. Rational functions: $r(x) = \dfrac{p(x)}{q(x)}, \quad q(x) \neq 0$
3. Radical functions: $f(x) = \sqrt[n]{x}$
4. Trigonometric functions: $\sin x, cox, \tan x, \csc x, \sec x, \cot x$

Explain why the following function is continuous at every point in its domain:

$$f(x) = x^2 + \cos x$$

The function $f(x) = x^2$ is a parabola and it is continuous at every real number and f(x) = cos x is continuous at every real number and therefore the sum of the two functions is continuous at every point in its domain

Continuity of a Composite function

If "g" is continuous at "c" and "f" is continuous at "c", then the composite function given by $(f \circ g) = f(g(x))$ is continuous at "c"

A) The following examples will define $f \circ g$ and determine all values of x for which $f \circ g$ is continuous .

Example #1: $f(x) = x^3, g(x) = \sqrt{x}$

$f \circ g = f(g(x)) = (\sqrt{x})^3$

$f \circ g = x^{3/2}$

domain $= \{x \geq 0\}$ or $[0, \infty)$

Example #2: $f(x) = \dfrac{1}{x-2}, g(x) = \sqrt{x}$

$f \circ g = f(g(x)) = \dfrac{1}{\sqrt{x} - 2}$

$f \circ g = \dfrac{\sqrt{x} + 2}{x - 4}$

domain= $[0,4) \cup (4, \infty)$

B) Determine the continuity of the following function

Example #1: $f(x) = \sin 3x$

$g(x) = 3x$

$f(x) = \sin x$

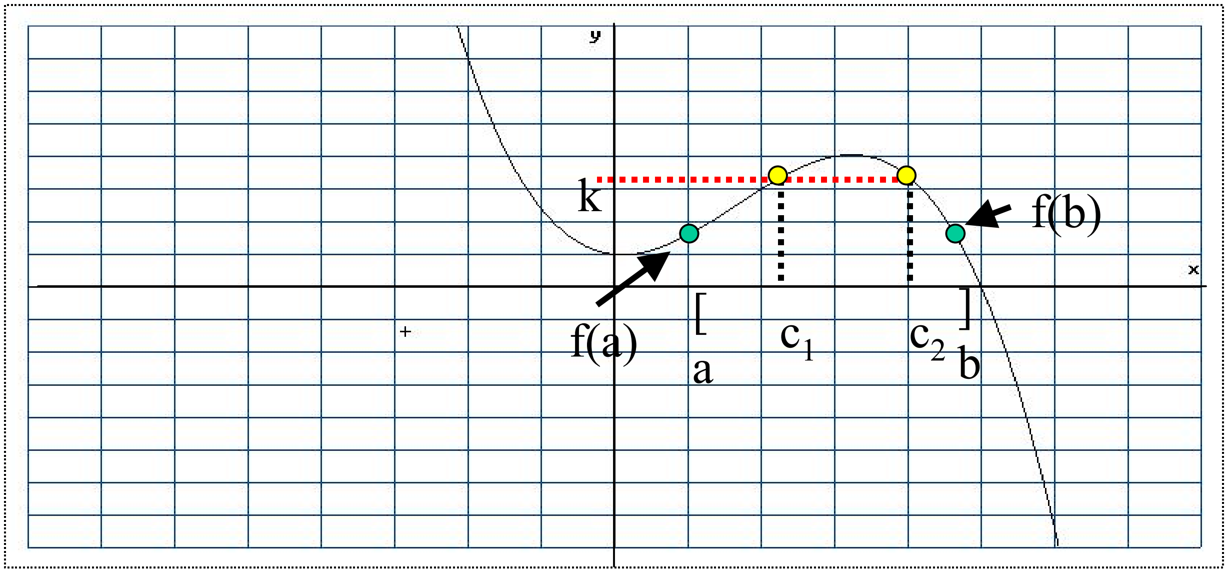
it follows that $f \circ g = f(g(x)) = \sin g(x) = \sin(3x)$

The Intermediate Value Theorem

If f is continuous on the closed interval [a, b] and k is any number between f (a) and f (b), then there is at least one number c in [a, b] such that f (c) = k

Note: a) this theorem informs you that a least one c exists but it does not give you the method for finding c

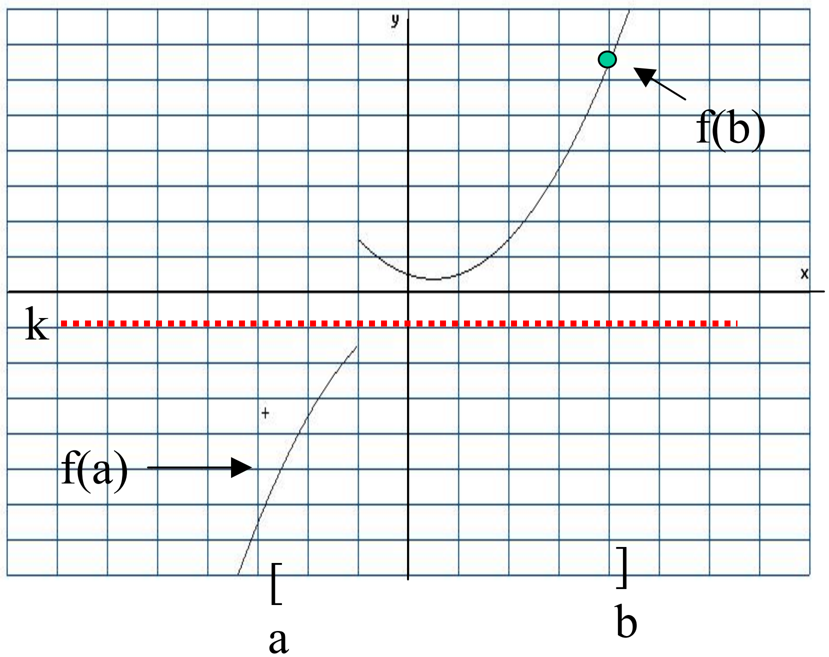
b) the theorem guarantees the existence of at least one number c in the closed interval but the possibility exists that there may more than one number c.



f is continuous on [a, b] for k there exists 2 c's)

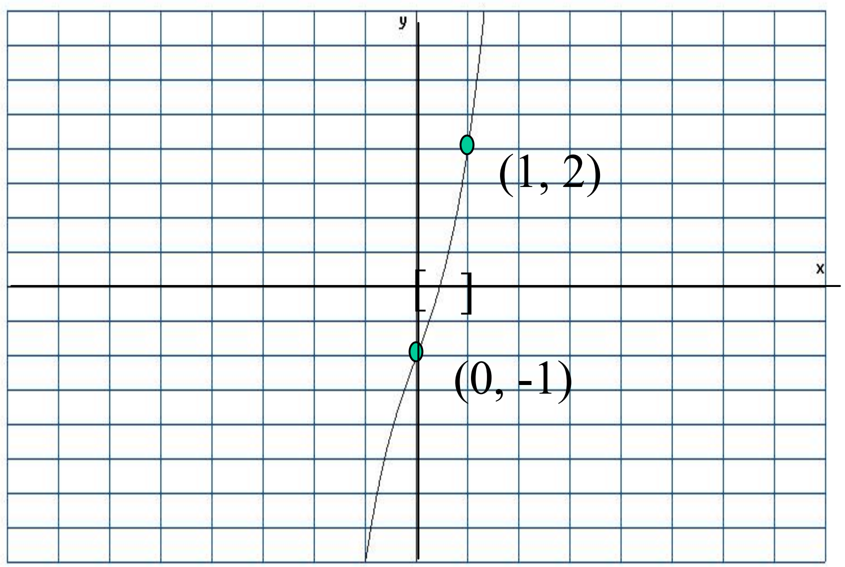
c) if a function is not continuous it may not contain an intermediate value

(f is not continuous on [a, b], for k there are no c’s)



d) can be used to locate the zeros of a function that is continuous on a closed interval if f (a) and f (b) differ in signs

(f is continuous on [0, 1] with f (0) < 0 and f (1) > 0)



Applications of the Intermediate Value Theorem

1. Show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval [0, 1]

f is continuous on the interval [0, 1] because

$$f(0) = 0^3 + 2(0) - 1 = -1 \text{ and } f(1) = 1^3 + 2(1) - 1 = 2$$

and as a result f (0) < 0 and f (1) > 0 and since the signs differ, the Intermediate Value Theorem can be applied and it can be concluded that there must be some c in [0, 1] such that f (c) = 0

---

2. Verify that the Intermediate Value Theorem applies to the polynomial function $f(x) = x^2 + x - 1$ over the indicated interval [0, 5] and find the value of c guaranteed by the theorem such that f (c) = 11.

f is continuous on the interval [0, 5] because

$$f(0) = 0^2 + 0 - 1 = -1 \text{ and } f(5) = 5^2 + 5 - 1 = 29$$

and because $f(0) \neq f(5)$ then we can conclude that f (c) takes on a value between f (a) and f (b), namely $f(x) = x^2 + x - 1 \Rightarrow 11 = c^2 + c - 1$

$$0 = c^2 + c - 12 \Rightarrow 0 = (x-3)(x+4) \Rightarrow x = 3 \text{ or } -4$$

The value c = 3 is selected because it exists in the interval [0, 5]

One-Sided Limits

Examination of the behavior of a limit as "x" approaches some "c" from some greater value from the right or as "x" approaches some "c" from some lesser value from the left.

$$\lim_{x \to c^+} f(x) = L$$ Limit from the right

$$\lim_{x \to c^-} f(x) = L$$ Limit from the left

Useful when:

a) taking the limits of functions involving radicals

b) investigation of the behavior of the step function

- one type is the greatest integer function [[x]] defined by

$[[x]] = \text{greatest integer } n \text{ such that } n \leq x$ for instance [[3.5]] = 3 and [[-3.5]] = -4

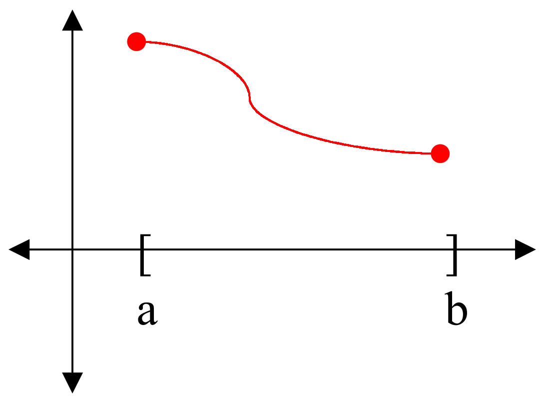
c) allows the extension of continuity to a closed interval. A function is continuous on a closed interval if it is continuous in the interior of the interval and possesses one-sided continuity at the endpoints

Definition on a Closed Interval

A function f is continuous on the closed interval [a, b] if it is continuous on the open interval (a, b)

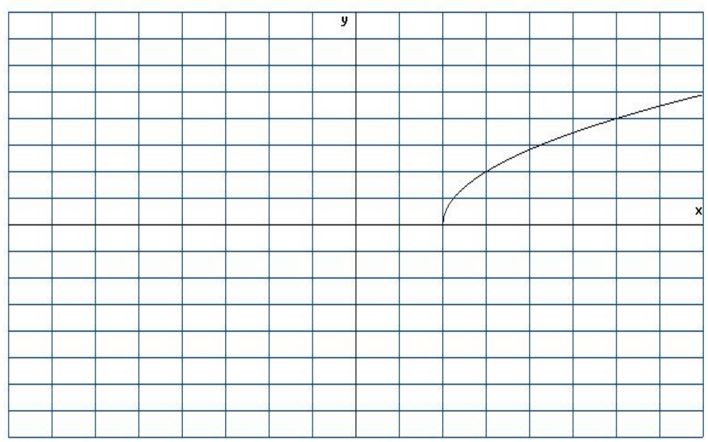
$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b)$$

The function f is continuous from the right at a and continuous from the left at b.(see figure at right)



Example #1

Find the limit of $f(x) = \sqrt{x-2}$ as x approaches 2 from the right



As indicated in the diagram the limit as a approaches 2 from the right is 0.

$$\lim_{x \to 2^+} \sqrt{x-2} = 0$$

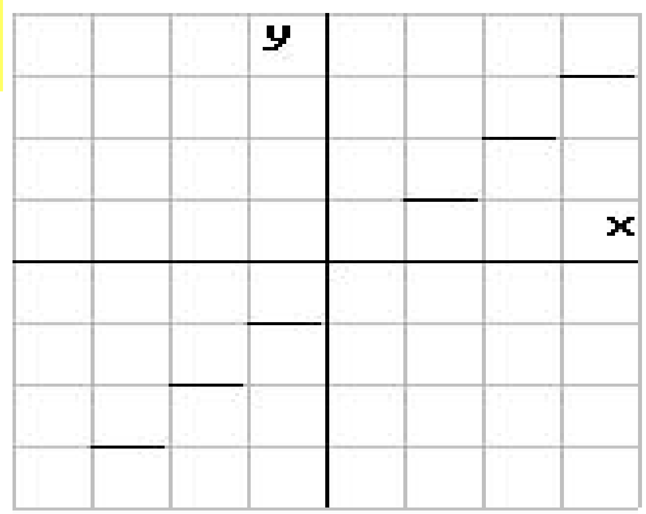
Example #2

Find the limit of $f(x) = [[x]]$ as x approaches 0 from the left and from the right

As indicated in the diagram, the limit as x approaches 0 from the left given by $\lim_{x \to 0^-} [[x]] = -1$

and the limit as x approaches 0 from the right is given by $\lim_{x \to 0^+} [[x]] = 0$

The greatest integer function is not continuous at 0 because left and right hand limits are different



Infinite Limits

A limit in which f(x) increases or decreases without bounds as x approaches some c is called an infinite limit. $\lim_{x \to c} f(x) = \infty$

You must remember that the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists. The correct interpretation is that it tells us that the limit fails to exist and that the function f(x) is showing unbounded behavior as x approaches c.

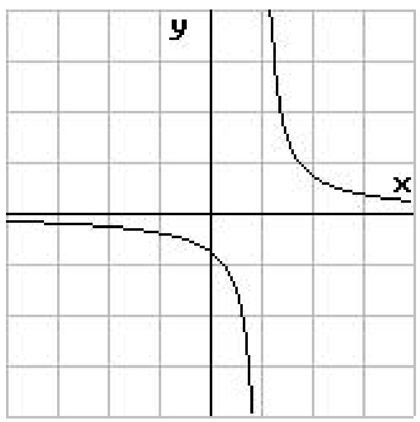
To help understand the concept lets us examine two functions

1.

$$f(x) = \frac{3}{x-1}$$

$$\lim_{x \to 1^+} \frac{3}{x-1} = \infty$$

$$\lim_{x \to 1^-} \frac{3}{x-1} = -\infty$$

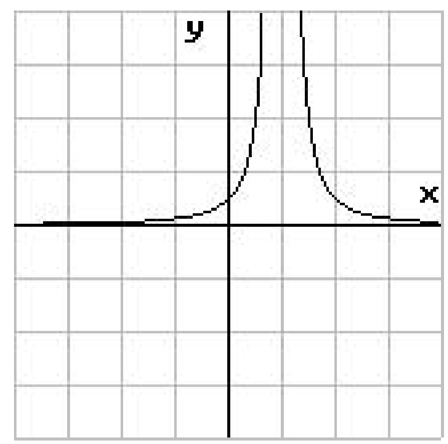


2.

$$f(x) = \frac{1}{(x-1)^2}$$

$$\lim_{x \to 1^+} \frac{1}{(x-1)^2} = \infty$$

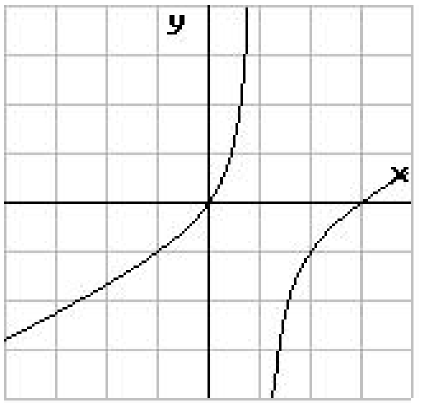
$$\lim_{x \to 1^-} \frac{1}{(x-1)^2} = \infty$$



To find an infinite limit

Find the limit of the following: $\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$ and $\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1}$

Using a graphing utility it is easy to determine the two answers



By inspection we can determine that:

1. as the graph approaches x =1 from the right the function decreases without bounds (minus infinity)
2. as the graph approaches x =1 from the left the function increases without bounds (positive infinity)

But what do we do if we do not use the graphing utility??

We know that the denominator is equal to zero when x = 1 and the numerator is not equal to zero and a vertical asymptote exists at x = 1. This means that each of the given limits is either positive or negative infinity. Now what??

Substitution: pick x values close to x = 1 from the right and from the left

example: from the right: x = 1.01 f (x) = -200.99 and x = 1.001 f (x) = -2000

conclusion as x approaches " 1" from the right the limit is negative infinity

from the left: x = 0.99 f (x) = 198.99 and x = 0.999 f (x) = 1999

conclusion as x approaches " 1" from the left the limit is positive infinity

.Properties of Infinite Limits

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L$$

1. Sum or difference: $\lim_{x\to c}[f(x) \pm g(x)] = \infty$

2. Product: $\lim_{x\to c}[f(x)g(x)] = \infty, \quad L > 0$

   $\lim_{x\to c}[f(x)g(x)] = -\infty, \quad L < 0$

3. Quotient: $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits.

Definition of a Vertical Asymptote

If f (x) approaches infinity (or negative infinity) as x approaches "d" from the right or the left, then the line x = d is a vertical asymptote of the graph of f.

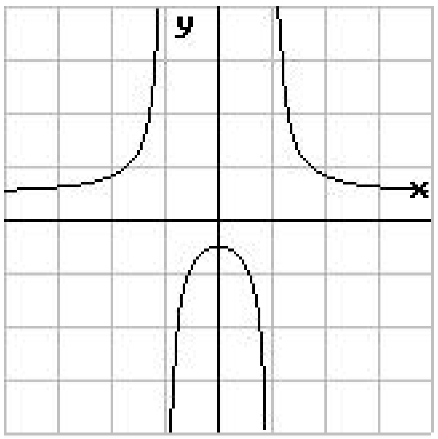
The Vertical Asymptotes Theorem states the following:

a) if the functions f and g are continuous on an open interval containing d and
b) if and $f(d) \neq 0$ and $g(d) = 0$
c) that for all $x \neq d$ $g(x) \neq 0$ then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$ has a vertical asymptote at x = d

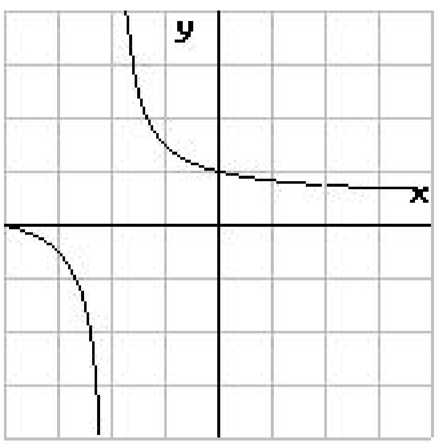
Determining Vertical Asymptotes

Example #1 $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$



By factoring the denominator you can see that the denominator is zero at x = -1 and x = 1. Also, the numerator is not zero at these two points. By application of the Vertical Asymptote Theorem you can conclude that there are two vertical asymptotes, namely x = -1 and x = 1.

Example #2: $f(x) = \dfrac{x^2 + 2x - 8}{x^2 - 4}$



$$f(x) = \frac{(x+4)(x-2)}{(x+2)(x-2)}$$

$$f(x) = \frac{(x+4)}{(x+2)}, x \neq 2$$

At all x-values other than x = 2, the graph coincides with the graph of f(x) = (x + 4)/(x + 2). The denominator is zero at x = -2 and the numerator is not zero at this point. Applying the Vertical Asymptotes Theorem it can be concluded that there is a vertical asymptote at x = -2.

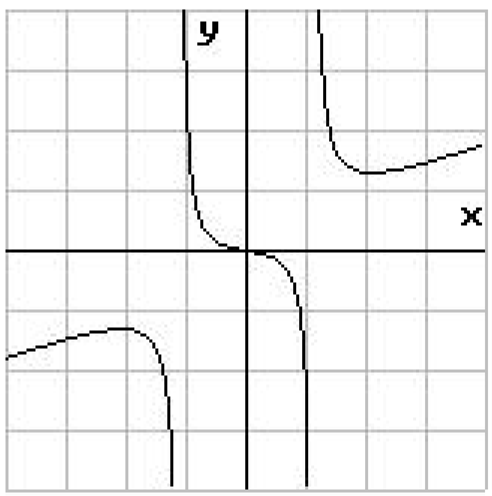
Note: there is no vertical asymptote at x = 2

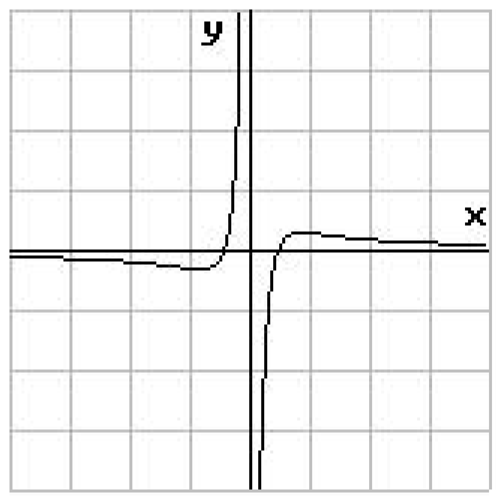
Limits at Infinity

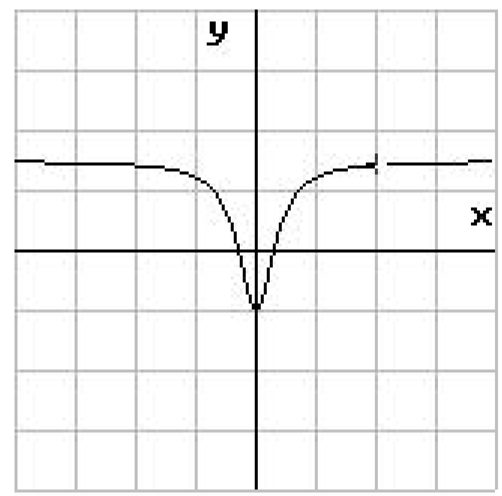
The focus of this section is to examine the "end behavior" of a function on an infinite interval. In simple terms we want to explore the behavior of a function when the values of x increase without bounds (positive infinity) or decrease without bounds (negative infinity)

The following three graphs illustrate the behavior of rational functions as the value of x approaches infinity:

$$f(x) = \frac{x^3 + 2x}{x^2 - 5} \qquad f(x) = \frac{x^2 - 5}{x^3 + 2x} \qquad f(x) = \frac{3x^2 - 1}{2x^2 + 1}$$







You can see that two of the graphs approach the line y = L as x increases without bounds. This line is referred to as a horizontal asymptote of the graph.

Definition of a Horizontal Asymptote:

The line y = L is a horizontal asymptote of the graph of f if

$$\lim_{x \to -\infty} f(x) = L \text{ or } \lim_{x \to \infty} f(x) = L$$

From this definition it follows that the graph of a function of x can have at most two horizontal asymptotes - one to the right and one to the left.

Limits at infinity have many of the same properties of limits as previously mentioned.

Example: if $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} g(x)$ both exist, then

$$\lim_{x \to \infty} [f(x) + g(x)] = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x)$$

$$\lim_{x \to \infty} [f(x) g(x)] = \left[\lim_{x \to \infty} f(x)\right]\left[\lim_{x \to \infty} g(x)\right]$$

For both ∞ and $-\infty$

A useful theorem: if r is any positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0, \quad \lim_{x \to +\infty} \frac{1}{x^r} = 0, \quad \lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Simple translation: a small number divided by a large number results is a value equal to 0.

Examples

$$\frac{4}{1{,}000{,}000{,}000} \approx 0, \quad \frac{30}{-1{,}000{,}000{,}000} \approx 0$$

To determine the limit at infinity

Example #1:

$$\lim_{x\to\infty}\frac{2x-1}{x+1}$$

$$\lim_{x\to\infty} 2x-1=\infty$$

$$\lim_{x\to\infty} x+1=\infty$$

This evaluation of the limit resulted in what is referred to as the indeterminate form ∞/∞

Example #2:

$$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}=\frac{\lim_{x\to\infty}\left(\frac{3x^2}{x^2}-\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(\frac{2x^2}{x^2}+\frac{1}{x^2}\right)}$$

To resolve the difficulty of the intermediate form rewrite the given expressions in an equivalent form by dividing the numerator and denominator by x^2. In general, when dividing use the highest power of x found in the denominator.

$$=\frac{\lim_{x\to\infty}\left(3-\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)}=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{1}{x^2}}$$

$$=\frac{3-0}{2+0}=\frac{3}{2}$$

Remember that a number divided by a very large number gives use a result of 0

Example #3:

$$\lim_{x\to\infty}\frac{x^2-5}{x^3+2x}$$

$$=\frac{\lim_{x\to\infty}\left[\frac{x^2}{x^3}-\frac{5}{x^3}\right]}{\lim_{x\to\infty}\left[\frac{x^3}{x^3}+\frac{2x}{x^3}\right]}$$

$$=\frac{\lim_{x\to\infty}\left[\frac{1}{x}-\frac{5}{x^3}\right]}{\lim_{x\to\infty}\left[1+\frac{2}{x^2}\right]}=\frac{\lim_{x\to\infty}\frac{1}{x}-\lim_{x\to\infty}\frac{5}{x^3}}{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{2}{x^2}}$$

$$=\frac{0-0}{1+0}=0$$

Example #4:

$$\lim_{x\to\infty}\frac{x^3+2x}{x^2-5}$$

$$=\frac{\lim_{x\to\infty}\left[\frac{x^3}{x^2}+\frac{2x}{x^2}\right]}{\lim_{x\to\infty}\left[\frac{x^2}{x^2}-\frac{5}{x^2}\right]}$$

$$=\frac{\lim_{x\to\infty}\left[x+\frac{2}{x}\right]}{\lim_{x\to\infty}\left[1-\frac{5}{x^2}\right]}=\frac{\lim_{x\to\infty}x+\lim_{x\to\infty}\frac{2}{x}}{\lim_{x\to\infty}1-\lim_{x\to\infty}\frac{5}{x^2}}$$

$$=\frac{\infty+0}{1-0}=\infty$$

Limit does not exist because the numerator increases without bound while the denominator approaches 1

Example #5. A function with two horizontal asymptotes

$$\lim_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}}$$

Remember for x > 0 we can write $x=\sqrt{x^2}$

$$\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}}$$

Remember for x < 0 we can write $x=-\sqrt{x^2}$

$$=\lim_{x\to\infty}\frac{3x-2}{(2x^2+1)^{\frac{1}{2}}}=\frac{\lim_{x\to\infty}\left(\frac{3x}{x}-\frac{2}{x}\right)}{\lim_{x\to\infty}\left(\frac{2x^2}{x^2}+\frac{1}{x^2}\right)^{\frac{1}{2}}}$$

$$=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{2}{x}}{\left(\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{1}{x^2}\right)^{\frac{1}{2}}}=\frac{3-0}{(2+0)^{\frac{1}{2}}}$$

$$=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

$$=\lim_{x\to-\infty}\frac{3x-2}{(2x^2+1)^{\frac{1}{2}}}=\frac{\lim_{x\to-\infty}\left(\frac{3x}{x}-\frac{2}{x}\right)}{\lim_{x\to-\infty}-\left(\frac{2x^2}{x^2}+\frac{1}{x^2}\right)^{\frac{1}{2}}}$$

$$=\frac{\lim_{x\to-\infty}3-\lim_{x\to-\infty}\frac{2}{x}}{-\left(\lim_{x\to-\infty}2+\lim_{x\to-\infty}\frac{1}{x^2}\right)^{\frac{1}{2}}}=\frac{3-0}{-(2+0)^{\frac{1}{2}}}$$

$$=\frac{3}{-\sqrt{2}}=-\frac{3\sqrt{2}}{2}$$

Note:

1. Dividing by x^2 inside a square root translates into dividing by x outside the square root.
2. Dividing by x^3 inside a square root translates into dividing by $x^{3/2}$ outside the square root
3. Dividing by x^3 inside a cube root means dividing by x outside the cube

If we compare the three graphs that were drawn and the results of the the calculations of limits as x approaches infinity the following conclusions can be drawn:

a) if the degree of the numerator is less than the degree of the dominator, the limit of the rational function is “0”.

b) if the degrees of the numerator and denominator are equal, the limit is the ratio of the two leading coefficients

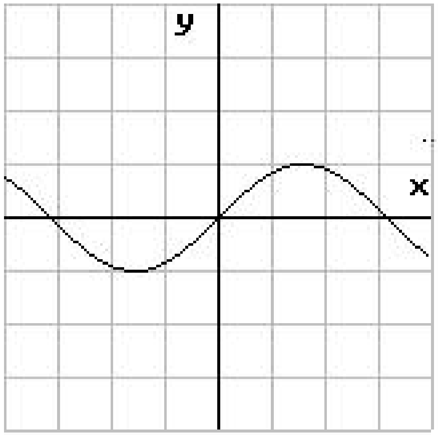
c) if the degree of the numerator is greater than that of the denominator, the limit does not exist

Limits of Trig Functions

We will confirm the the limits of certain trig functions in an informal graphical analysis format. We assume that each variable represents a real number or a radian measure of an angle

Graph:

$f(x) = \sin x$



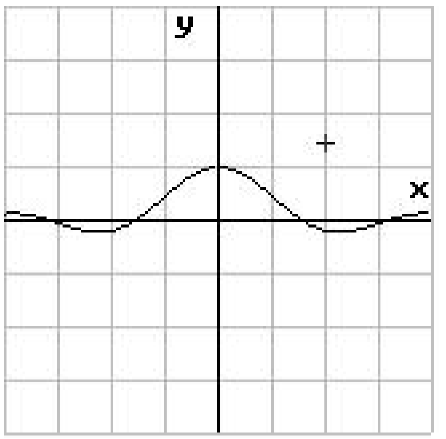
By inspection you can see that as x approaches 0 from from both the left and right the value of f(x) = 0

Therefore:

$$\lim_{x \to 0} \sin x = 0$$

Graphs:

$$f(x) = \frac{\sin x}{x}$$

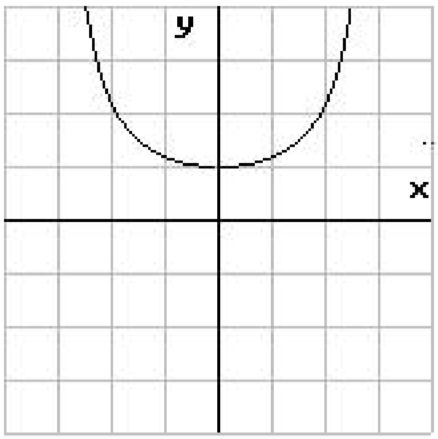


By inspection you can see that as x approaches 0 from from both the left and right the value of f(x) = 1

Therefore:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$f(x) = \frac{x}{\sin x}$$

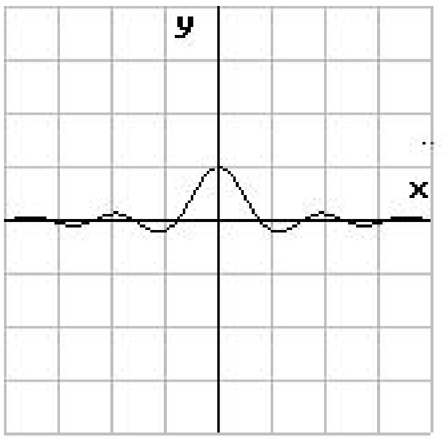


By inspection you can see that as x approaches 0 from from both the left and right the value of f(x) = 1

Therefore:

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

$$f(x) = \frac{\sin(4x)}{(4x)}$$



By inspection you can see that as x approaches 0 from from both the left and right the value of f(x) = 1

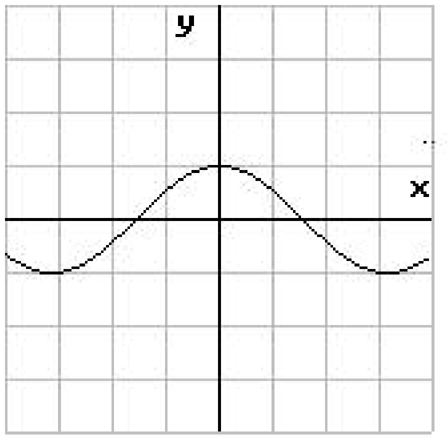
Therefore:

$$\lim_{x \to 0} \frac{\sin(ax)}{(ax)} = 1$$

$$\lim_{x \to 0} \frac{ax}{\sin(ax)} = 1$$

Graphs:

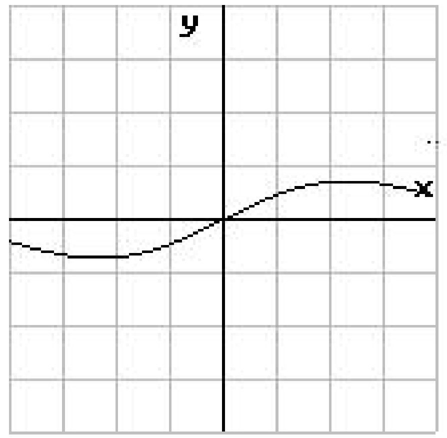
$$f(x) = \cos x$$



By inspection you can see that as x approaches 0 from from both the left and right the value of f(x) = 1
Therefore:

$$\lim_{x \to 0} \cos x = 1$$

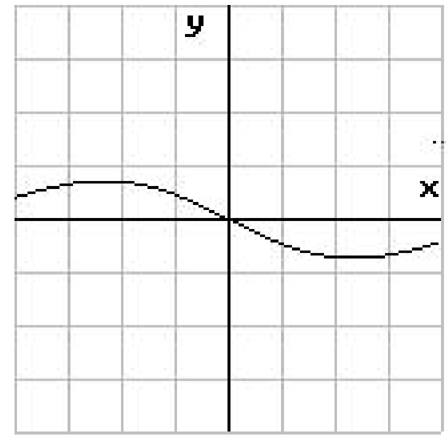
$$f(x) = \frac{1 - \cos x}{x}$$



By inspection you can see that as x approaches 0 from from both the left and right the value of f(x) = 0
Therefore:

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

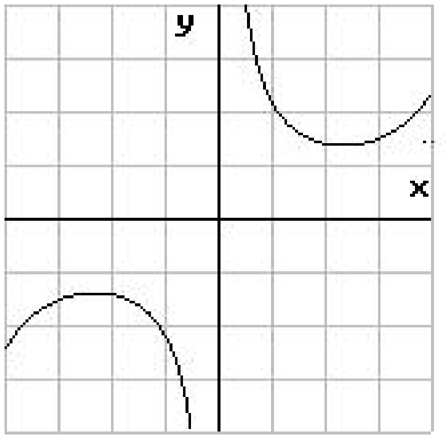
$$f(x) = \frac{\cos x - 1}{x}$$



By inspection you can see that as x approaches 0 from both the left and right the value of f(x) = 0
Therefore:

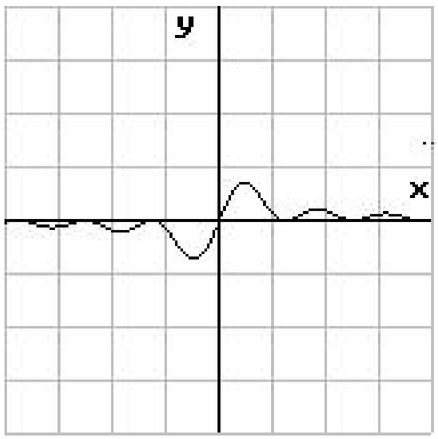
$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Graphs: $f(x) = \dfrac{x}{1-\cos x}$



By inspection you can see that as x approaches 0 from from the left the graph decreases without bounds and from the right the graph increases without bounds. Therefore the limit of the reciprocal does not exist

$f(x) = \dfrac{1-\cos(5x)}{(5x)}$



By inspection you can see that as x approaches 0 from from both the left and right the value of f(x) = 0 Therefore:

$$\lim_{x\to 0}\frac{1-\cos ax}{ax} = 0 \qquad \lim_{x\to 0}\frac{\cos ax - 1}{ax} = 0$$

Examples

Procedures:

1. Use direct substitution first
2. Manipulate the expression (common denominators, factoring, expansion and trig substitutions) so that one of the trig limit laws can be applied.
3. Simplify the answer

1. $\lim_{x\to 0}\frac{\sin^3 x}{(2x)^3}=\lim_{x\to 0}\left(\frac{\sin x}{2x}\cdot\frac{\sin x}{2x}\cdot\frac{\sin x}{2x}\right)=\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{2}\cdot 1\cdot\frac{1}{2}\cdot 1\cdot\frac{1}{2}\cdot 1=\frac{1}{8}$

2. $\lim_{x\to 0}\frac{2\cos x-2}{3x}=\lim_{x\to 0}\frac{2(\cos x-1)}{3x}=\frac{2}{3}\lim_{x\to 0}\frac{(\cos x-1)}{x}=\frac{2}{3}\cdot 0=0$

3. $\lim_{x\to 0}\frac{4x^2+3x\sin x}{x^2}=\lim_{x\to 0}\left(\frac{4x^2}{x^2}+\frac{3x}{x}\cdot\frac{\sin x}{x}\right)=\lim_{x\to 0}\left(4+3\frac{\sin x}{x}\right)=4+3(1)=7$

4. $\lim_{x\to 0}\frac{x\sin x}{x^2+1}=\frac{0\cdot 0}{0^2+1}=\frac{0}{1}=0$

5. $\lim_{x\to 0}\frac{\cos x}{1-\sin x}=\lim_{x\to 0}\frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)}=\lim_{x\to 0}\frac{\cos x(1+\sin x)}{1-\sin^2 x}=\lim_{x\to 0}\frac{\cos x(1+\sin x)}{\cos^2 x}$

$=\lim_{x\to 0}\frac{(1+\sin x)}{\cos x}=\frac{1+0}{1}=1$

6. $\lim_{x\to 0}\frac{1-2x^2-2\cos x+\cos^2 x}{x^2}=\lim_{x\to 0}\left(\frac{-2x^2+(1-\cos x)^2}{x^2}\right)=$

$\lim_{x\to 0}\left(\frac{-2x^2}{x^2}+\frac{(1-\cos x)}{x}\cdot\frac{(1-\cos x)}{x}\right)=-2+0\cdot 0=-2$

7. $\lim_{x\to 0}\frac{x+\tan x}{\sin x} = \lim_{x\to 0}\left(\frac{x+\frac{\sin x}{\cos x}}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\frac{x\cos x+\sin x}{\cos x}}{\sin x}\right) = \lim_{x\to 0}\left(\frac{x\cos x+\sin x}{\sin x\cos x}\right) =$

$$\lim_{x\to 0}\left(\frac{x\cos x}{\sin x\cos x}+\frac{\sin x}{\sin x\cos x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin x}+\frac{1}{\cos x}\right) = 1+1=0$$

Applications of Limits

1. Tangent Lines

required information: slope formula $m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

slope intercept formula $(y_2 - y_1) = m(x_2 - x_1)$

Example #1: determine the equation of the line tangent to the curve $f(x)=3x^3-2x+7$ at the point (1, 8)

A) Slope Formula

$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[3(x+h)^3-2(x+h)+7]-[3x^3-2x+7]}{h}=$$

$$\lim_{h\to 0}\frac{[3x^3+9x^2h+9xh^2+3h^3-2x-2h+7]-[3x^3-2x+7]}{h}=$$

$$\lim_{h\to 0}\frac{h[9x^2+9xh+3h^2-2]}{h}=\lim_{h\to 0}9x^2+9xh+3h^2-2=9x^2-2$$

B. Slope = m = $9x^2 - 2 = 9(1)^2 - 2 = 9 - 2 = 7$

C. Equation $(y_2 - y_1) = m(x_2 - x_1) \rightarrow (y - 8) = 7(x - 1) \rightarrow y = 7x + 1$

Example #2: Find the slope of the tangent lines to the graph of the function $f(x) = \sqrt{2x - 3}$ at the points (6, 3) and (14, 5)

A. Slope Formula:

$$m = \lim_{h \to 0} \frac{\sqrt{2(x + h) - 3} - \sqrt{2x - 3}}{h} =$$

$$\lim_{h \to 0} \frac{\sqrt{2(x + h) - 3} - \sqrt{2x - 3}}{h} \cdot \frac{\sqrt{2(x + h) - 3} + \sqrt{2x - 3}}{\sqrt{2(x + h) - 3} + \sqrt{2x - 3}} =$$

$$\lim_{h \to 0} \frac{[2(x + h) - 3] - (2x - 3)}{h} =$$

$$\lim_{h \to 0} \frac{2x + 2h - 3 - 2x + 3}{h(\sqrt{2(x + h) - 3} + \sqrt{2x - 3})} = \lim_{h \to 0} \frac{2h}{h(\sqrt{2(x + h) - 3} + \sqrt{2x - 3})}$$

$$= \frac{2}{\sqrt{2x - 3} + \sqrt{2x - 3}} = \frac{2}{2\sqrt{2x - 3}} = \frac{1}{\sqrt{2x - 3}}$$

B. at the point (6, 3) the slope is $m = \frac{1}{\sqrt{2x - 3}} = \frac{1}{\sqrt{2(6) - 3}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

and at the point (14, 5) the slope is $m = \frac{1}{\sqrt{2x - 3}} = \frac{1}{\sqrt{2(14) - 3}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$

2. Velocity

The function f that describes the motion of an object is called the position function of the object. In the time interval from t = x to t = x + h, the change in position is f(x + h) - f(x). The average velocity over this time period is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(x+h)-f(x)}{h}$$

When average velocities are considered over shorter and shorter periods of time we define instantaneous velocity (limit of average velocities) as

$$v(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Example

A ball is thrown vertically upward from the ground with an initial velocity of 64feet/sec. The equation of motion is $s(t) = -16t^2 + 64t$

If " t' is the number of seconds that has elapsed since the ball was thrown, and " s" is the number of meters the ball has traveled from the starting point at " t" seconds, find:

1. the instantaneous velocity of the ball at the end of I second
2. the instantaneous velocity of the ball at the end of 3 seconds
3. how many seconds it takes the ball to reach its highest point
4. how high will the ball go
5. the speed of the ball at the end of 1 second and at the end of 3 seconds
6. how many seconds will it take to reach the ground
7. the instantaneous velocity of the ball when it reaches the ground

Before staring to find the answer for each part of the question we need to determine the velocity formula.

$$v = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[-16(t+h)^2 + 64(t+h)] - [-16t^2 + 64t]}{h} =$$

$$\lim_{h \to 0} \frac{[-16t^2 - 32ht - 16h^2 + 64t + 64h] - [-16t^2 + 64t]}{h} =$$

$$\lim_{h \to 0} \frac{-16t^2 - 32ht - 16h^2 + 64t + 64h + 16t^2 - 64t}{h} =$$

$$\lim_{h \to 0} \frac{h(-32t - 16h + 64)}{h} = -32t + 64$$

1 .v = -32t + 64 = -32(1) + 64 = 32; so that at the end of 1 sec the ball is rising at 32ft/sec

2. v = -32t + 64 = -32(3) + 64 = -32; so that at the end of 3 sec the ball is falling at 32 ft/sec

3. The ball reaches its highest point when the direction of motion changes or when the velocity is equal to zero. v = -32t + 64, 0 = -32t + 64, t = 2

4. When t = 2, s (t) = $-16(2)^2+64(2)$ = -64 +128 = 64, therefore the ball reaches the highest point 64 feet above the starting point

5. The speed of the ball at particular time periods is equal to the absolute value of the velocity at the those time periods: t = 1 |32| = 32 and t = 3 |-32| = 32

6. The ball will reach the ground when s (t) = 0. Setting s (t) to zero and solving

$$0 = -16t^2 + 64t \Rightarrow 0 = -16t(t-4) \Rightarrow t = 0, t = 4$$

Therefore, the ball will reach the ground in 4 sec.

7. v= -32t + 64 = -32(4) + 64 = -128 + 64 = -64; when the ball reaches the ground, its instantaneous velocity is -64 ft/sec

3. Other rate of change

Let us consider the situation where y is a quantity that is dependent on another quantity x or in simpler terms "y is a function of x". If x changes from x_1 to x_2, then the change in x (increment) is $\Delta x = x_2 - x_1$ and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$

The difference quotient $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called the average rate of change of y with respect to x. If the average rate of change in x is over smaller and smaller intervals Δx approaches 0. The limit of these averages is called the instantaneous rate of change of y with respect to x.

$$\text{Instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x2 \to x1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example #1: The displacement in meters of a particle moving in a straight line is given by $s(t) = 4t^2 - 5t + 12$ where t is measured in seconds. Find the average velocity over the time interval [2, 4]

$$s(2) = 4(2)^2 - 5(2) + 12 = 18$$

$$s(4) = 4(4)^2 - 5(4) + 12 = 56$$

$$av = \frac{\Delta y}{\Delta x} = \frac{18 - 56}{2 - 4} = \frac{-38}{-2} = 19$$

Example #2: The cost (in dollars) of producing x units of a certain commodity is $c(x) = 4500 + 20x + 0.08x^2$ Find the average rate of change of "c" with respect to x when the production level is changed from 200 to 210.

$$c(200) = 4500 + 20(200) + 0.08(200)^2 = 11700$$

$$c(200) = 4500 + 20(210) + 0.08(210)^2 = 12228$$

$$average_{\cos t} = \frac{\Delta c}{\Delta x} = \frac{11700 - 12228}{200 - 210} = \frac{-528}{-10} = 52.8$$