GEOMETRIC SUMS

1. Find the indicated sum:

a) 2, 6, 18, ... to 5 terms

$a = 2, r = 3, n = 5$

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \frac{2(1-3^5)}{(1-3)} \Rightarrow S_n = \frac{2(1-243)}{(-2)} \Rightarrow S_n = \frac{2(-242)}{(-2)} = 242$$

b) -1, 3, -9, ... to 6 terms

$a = -1, r = -3, n = 6$

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \frac{-1(1-(-3)^6)}{(1-(-3))} \Rightarrow S_n = \frac{-1(1-729)}{(4)} \Rightarrow S_n = \frac{-1(-728)}{(4)} = 182$$

c) -2, -10, - 50, ... to 6 terms

$a = -2, r = 5, n = 6$

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \frac{-2(1-(5)^6)}{(1-5)} \Rightarrow S_n = \frac{-2(1-15625)}{(-4)} \Rightarrow S_n = \frac{-2(-15624)}{(-4)} = -7812$$

d) 3/16, 3/4, 3, ...to 7 terms

$a = 3/16, r = 4, n = 7$

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \frac{3/16\,(1-(4)^7)}{(1-4)} \Rightarrow S_n = \frac{3(1-(16384))}{16(-3)} \Rightarrow S_n = \frac{3(-16383)}{16(-3)} = \frac{16383}{16}$$

e) a = 4, l = 324, r = 3

$$S_n = \frac{a-rl}{1-r} \Rightarrow S_n = \frac{4-3\cdot 324}{1-3} \Rightarrow S_n = \frac{4-972}{-2} = \frac{-968}{-2} = 484$$

f) a = 64, r = -1/4, l = 1/4

$$S_n = \frac{a-rl}{1-r} \Rightarrow S_n = \frac{64-\left(-1/4\right)\cdot 1/4}{1-\left(-1/4\right)} \Rightarrow S_n = \frac{64+1/16}{5/4} = \frac{1025/16}{5/4} = \frac{1025\cdot 4}{5\cdot 16} = \frac{205}{4}$$

g) a = 4, l = 324, r = -3

$$S_n = \frac{a-rl}{1-r} \Rightarrow S_n = \frac{4-(-3)\cdot 324}{1-(-3)} \Rightarrow S_n = \frac{4+972}{4} = \frac{976}{4} = 244$$

h) 1000, 100, 10, ...; n = 7

$r = 100/1000 = 1/10$

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \frac{1000\left(1-\left(1/10\right)^7\right)}{\left(1-\left(1/10\right)\right)} \Rightarrow S_n = \frac{1000\left(1-\left(1/10000000\right)\right)}{\left(1-\left(1/10\right)\right)} \Rightarrow$$

$$S_n = \frac{1000\left(9999999/10000000\right)}{\left(9/10\right)} = \frac{9999999/10000}{9/10} = \frac{9999999\cdot 10}{9\cdot 10000} = \frac{1111111}{1000} = 1111.111$$

2. Find the two values that are not given:

a) a = 8, r = -3, n = 5

$$l = ar^{n-1} \Rightarrow l = 8\cdot(-3)^{5-1} \Rightarrow l = 8\cdot(-3)^{4} \Rightarrow l = 8\cdot 81 = 648$$

$$S_n = \frac{a-rl}{1-r} \Rightarrow S_n = \frac{8-(-3)648}{1-(-3)} \Rightarrow S_n = \frac{8+1944}{4} = \frac{1952}{4} = 488$$

b) a = 3, l = 48, S_n = 33

$$S_n = \frac{a-rl}{1-r} \Rightarrow 33 = \frac{3-(r)48}{1-r} \Rightarrow 33(1-r) = 3-48r \Rightarrow 33-33r = 3-48r \Rightarrow 30 = -15r \Rightarrow r = -2$$

$$l = ar^{n-1} \Rightarrow 48 = 3\cdot(-2)^{n-1} \Rightarrow 16 = (-2)^{n-1} \Rightarrow \left(-2\right)^{4} = (-2)^{n-1} \Rightarrow n = 5$$

c) 1
225, r = 5, n = 5

$$l = ar^{n-1} \Rightarrow 225 = a\cdot(5)^{5-1} \Rightarrow 225 = a\cdot(5)^{4} \Rightarrow 225 = 625a \Rightarrow a = {}^{225}\!/_{625} = {}^{9}\!/_{25}$$

$$S_n = \frac{a-rl}{1-r} \Rightarrow S_n = \frac{{}^{9}\!/_{25}-(5)225}{1-(5)} \Rightarrow S_n = \frac{{}^{9}\!/_{25}-1125}{-4} = \frac{{}^{-28116}\!/_{25}}{4} = {}^{-28116}\!/_{4\cdot 25} = {}^{-7029}\!/_{25}$$

d) a = 3, n = 3, S_n = 19/3

$$S_n = \frac{a\left(1-r^{n}\right)}{\left(1-r\right)} \Rightarrow \frac{19}{3} = \frac{3\left(1-r^{3}\right)}{\left(1-r\right)} \Rightarrow \frac{19}{3} = \frac{3\left(1-r\right)(1+r+r^{2})}{\left(1-r\right)} \Rightarrow \frac{19}{3} = 3(1+r+r^{2}) \Rightarrow$$

$$\frac{19}{3} = 3+3r+3r^{2} \Rightarrow 19 = 9+9r+9r^{2} \Rightarrow 9r^{2}+9r-10 = 0 \Rightarrow (3r+5)(3r-2) = 0$$

$$r = {}^{-5}\!/_{3}, r = {}^{2}\!/_{3}$$

$$l = ar^{n-1} \Rightarrow l = 3\left({}^{-5}\!/_{3}\right)^{3-1} \Rightarrow l = 3\left({}^{-5}\!/_{3}\right)^{2} \Rightarrow l = 3\left({}^{25}\!/_{9}\right) = {}^{25}\!/_{3}$$

$$\text{or }\; l = ar^{n-1} \Rightarrow l = 3\left(2-{}^{5}\!/_{3}\right)^{3-1} \Rightarrow l = 3\left({}^{2}\!/_{3}\right)^{2} \Rightarrow l = 3\left({}^{4}\!/_{9}\right) = {}^{4}\!/_{3}$$

e) a = 5/9, r = -3, S_n = -100/9

$$S_n = \frac{a-rl}{1-r} \Rightarrow \frac{-100}{9} = \frac{{}^{5}\!/_{9}-(-3)l}{1-(-3)} \Rightarrow \frac{-100}{9} = \frac{{}^{5}\!/_{9}+3l}{4} \Rightarrow \frac{-100}{9} = \frac{\frac{5+27l}{9}}{4} \Rightarrow$$

$$\frac{-100}{9} = \frac{5+27l}{36} \Rightarrow -3600 = 45+243l \Rightarrow -3645 = 243l \Rightarrow l = {}^{-135}\!/_{9}$$

$$l = ar^{n-1} \Rightarrow {}^{-135}\!/_{9} = {}^{5}\!/_{9}\cdot(-3)^{n-1} \Rightarrow {}^{-135}\!/_{9}\cdot{}^{9}\!/_{5} = (-3)^{n-1} \Rightarrow -27 = (-3)^{n-1} \Rightarrow (-3)^{3} = (-3)^{n-1} \Rightarrow n = 4$$

f) a = -56, l = 7/4, n = 6

$$l = ar^{n-1} \Rightarrow {}^{7}\!/_{4} = -56\cdot(r)^{6-1} \Rightarrow \frac{7}{4\cdot -56} = (r)^{5} \Rightarrow \frac{1}{-32} = r^{5} \Rightarrow \left(\frac{-1}{2}\right)^{5} = r^{5} \Rightarrow r = \frac{-1}{2}$$

$$S_n = \frac{a-rl}{1-r} \Rightarrow S_n = \frac{-56-\left({}^{-1}\!/_{2}\right){}^{7}\!/_{4}}{1-\left({}^{-1}\!/_{2}\right)} \Rightarrow S_n = \frac{-56+{}^{7}\!/_{8}}{{}^{3}\!/_{2}} = \frac{{}^{-441}\!/_{8}}{{}^{3}\!/_{2}} = \frac{-441\cdot 2}{3\cdot 8} = \frac{-147}{4}$$

f) $n = 9, r = 2, S_n = 1022$

$$S_n = \frac{a\left(1-r^n\right)}{\left(1-r\right)} \Rightarrow 1022 = \frac{a\left(1-(2)^9\right)}{\left(1-2\right)} \Rightarrow 1022 = \frac{a\left(1-512\right)}{\left(-1\right)} \Rightarrow -1022 = -511a \Rightarrow a = 2$$

$$l = ar^{n-1} \Rightarrow l = 2\left(2\right)^{9-1} \Rightarrow l = 2\left(2\right)^8 \Rightarrow l = 3\left(256\right) = 512$$

i) $a = -2, n = 3, S_n = -14$

$$S_n = \frac{a\left(1-r^n\right)}{\left(1-r\right)} \Rightarrow -14 = \frac{-2\left(1-r^3\right)}{\left(1-r\right)} \Rightarrow -14 = \frac{-2\left(1-r\right)(1+r+r^2)}{\left(1-r\right)} \Rightarrow -14 = -2(1+r+r^2) \Rightarrow$$

$$-14 = -2-2r-2r^2 \Rightarrow 2r^2+2r-12 = 0 \Rightarrow (r+4)(r-3) = 0$$

$$r = -4, r = 3$$

$$l = ar^{n-1} \Rightarrow l = -2\left(-4\right)^{3-1} \Rightarrow l = -2\left(-4\right)^2 \Rightarrow l = -2\left(16\right) = -32$$

$$\text{or } \ l = ar^{n-1} \Rightarrow l = -2\left(3\right)^{3-1} \Rightarrow l = -2\left(3\right)^2 \Rightarrow l = -2\left(9\right) = -18$$

j) $a = -2, n = 3, S_n = -1302$

$$S_n = \frac{a\left(1-r^n\right)}{\left(1-r\right)} \Rightarrow -1302 = \frac{-2\left(1-r^3\right)}{\left(1-r\right)} \Rightarrow -1302 = \frac{-2\left(1-r\right)(1+r+r^2)}{\left(1-r\right)} \Rightarrow -1302 = -2(1+r+r^2) \Rightarrow$$

$$-1302 = -2-2r-2r^2 \Rightarrow 2r^2+2r-1300 = 0 \Rightarrow (r+26)(r-25) = 0$$

$$r = -26, r = 26$$

$$l = ar^{n-1} \Rightarrow l = -2\left(-26\right)^{3-1} \Rightarrow l = -2\left(-26\right)^2 \Rightarrow l = -2\left(676\right) = -1352$$

$$\text{or } \ l = ar^{n-1} \Rightarrow l = -2\left(25\right)^{3-1} \Rightarrow l = -2\left(25\right)^2 \Rightarrow l = -2\left(625\right) = -1250$$

k) $a = 17, r = -1/2, S_n = 187/16$

$$S_n = \frac{a-rl}{1-r} \Rightarrow \frac{187}{16} = \frac{17-(-\frac{1}{2})l}{1-\left(-\frac{1}{2}\right)} \Rightarrow \frac{187}{16} = \frac{17+\frac{1}{2}l}{1+\frac{1}{2}} \Rightarrow \frac{187}{16} = \frac{34+l/2}{3/2} \Rightarrow \frac{187}{16} = \frac{34+l}{3} \Rightarrow$$

$$561 = 544+16l \Rightarrow 16l = 17 \Rightarrow l = \tfrac{17}{16}$$

$$l = ar^{n-1} \Rightarrow \tfrac{17}{16} = 17\cdot(-\tfrac{1}{2})^{n-1} \Rightarrow \frac{17}{16\cdot 17} = (-\tfrac{1}{2})^{n-1} \Rightarrow \tfrac{1}{16} = (-\tfrac{1}{2})^{n-1} \Rightarrow$$

$$\left(-\tfrac{1}{2}\right)^4 = (-\tfrac{1}{2})^{n-1} \Rightarrow n = 5$$

3. a) The fourth and eighth terms of a sequence of positive numbers in a G.P. are 1/4 and 4 respectively. Find the fifth number.

Mini Sequence

$$a = \tfrac{1}{4}, l = 4, n = 5$$

$$l = ar^{n-1} \Rightarrow 4 = \tfrac{1}{4}\cdot r^{5-1} \Rightarrow 16 = r^4 \Rightarrow r = \pm 2$$

$$\text{4th term times the ratio } \ \tfrac{1}{4}\cdot \pm 2 = \pm\tfrac{2}{4} = \pm\tfrac{1}{2}$$

b) The third term of a G.P. is 5 and the sixth term is $8/\sqrt{5}$. Find the intervening terms.

Mini Sequence

$a = 5, l = 8/\sqrt{5}, n = 4$

$l = ar^{n-1} \Rightarrow 8/\sqrt{5} = 5r^{4-1} \Rightarrow 8/\sqrt{5} = 5r^3 \Rightarrow 8/5\sqrt{5} = r^3 \Rightarrow \frac{2^3}{5^{\frac{3}{2}}} = r^3 \Rightarrow$

$\frac{(2^3)^{\frac{1}{3}}}{\left(5^{\frac{3}{2}}\right)^{\frac{1}{3}}} = (r^3)^{\frac{1}{3}} \Rightarrow r = \frac{2}{5^{\frac{1}{2}}}$

$5, 5 \cdot \frac{2}{5^{\frac{1}{2}}} = \underline{5^{\frac{1}{2}} \cdot 2}, 5^{\frac{1}{2}} \cdot 2 \cdot \frac{2}{5^{\frac{1}{2}}} = \underline{4}, 8/\sqrt{5}$

c) The fourth term of a G.P. is 2 and the seventh term is -2. Find the intervening terms.

Mini Sequence

$a = 2, l = -2, n = 4$

$l = ar^{n-1} \Rightarrow -2 = 2 \cdot r^{4-1} \Rightarrow -2 = 2 \cdot r^3 \Rightarrow -1 = r^3 \Rightarrow r = -1$

$2, 2 \cdot -1 = \underline{-2}, -2 \cdot -1 = \underline{2}, -2$

d) The product of three real numbers is G.P. is -64. The first is 4 times the third. Find the numbers.

$a \cdot ar \cdot ar^2 = -64 \Rightarrow a^3r^3 = -64$

$a = 4 \cdot ar^2 \Rightarrow 1/4 = r^2 \Rightarrow r = \pm\frac{1}{2}$ substitute into first equation

$a^3\left(\pm\frac{1}{2}\right)^3 = -64 \Rightarrow a^3\left(\pm\frac{1}{8}\right) = -64 \Rightarrow a^3 = \pm 512 \Rightarrow a = \pm 8$

$\pm 8, \quad \pm 8 \cdot \pm\frac{1}{2} = 4, \quad 4 \cdot \pm\frac{1}{2} = \pm 2$

e) Find the first term of a geometric progression whose common ratio is 2 and whose sixth term is 96.

$l = 96, r = 2, n = 6$

$l = ar^{n-1} \Rightarrow 96 = a \cdot 2^{6-1} \Rightarrow 96 = a \cdot 2^5 \Rightarrow 96 = a \cdot 32 \Rightarrow a = 3$

f) Find the first term in a geometric progression whose common ratio is 3 and whose fifth term is 324.

$l = 324, r = 3, n = 5$

$l = ar^{n-1} \Rightarrow 324 = a \cdot 3^{5-1} \Rightarrow 324 = a \cdot 3^4 \Rightarrow 324 = a \cdot 81 \Rightarrow a = 4$

g) The sum of the first 8 terms of a geometric series is 17 times the sum of its first four terms. Find the common ratio

$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow$ sum of 1st 8 terms $= S_n = \frac{a(1-r^8)}{(1-r)}$, sum of the 1st 4 terms $= S_n = \frac{a(1-r^4)}{(1-r)}$

$\frac{a(1-r^8)}{(1-r)} = 17\left[\frac{a(1-r^4)}{(1-r)}\right] \Rightarrow (1-r^8) = 17(1-r^4) \Rightarrow (1-r^4)(1+r^4) = 17(1-r^4) \Rightarrow$

$(1+r^4) = 17 \Rightarrow r^4 = 16 \Rightarrow r = \pm 2$

h) In a lottery, the first ticket drawn paid a prize of $30,000. Each succeeding ticket paid half as much as the preceding one. If six tickets were drawn, what is the total prize money paid?

$$a = 30{,}000, r = \tfrac{1}{2}, n = 6$$

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \frac{30000\left(1-\tfrac{1}{2}^6\right)}{\left(1-\tfrac{1}{2}\right)} \Rightarrow S_n = \frac{30000\left(1-\tfrac{1}{64}\right)}{\left(\tfrac{1}{2}\right)} \Rightarrow$$

$$S_n = \frac{30000\left(\tfrac{63}{64}\right)}{\left(\tfrac{1}{2}\right)} \Rightarrow S_n = \frac{30000 \cdot 63 \cdot 2}{64} \Rightarrow S_n = 59{,}562.50$$

i) The value of a certain rare coin increases 10% each year. If the coin is worth $3.00 now, what is its approximate value in 5 years?

$$a = 3, r = 1.1, n = 6 \text{ (now plus five years)}$$

$$l = ar^{n\text{-}1} \Rightarrow l = 3\cdot(1.1)^{6-1} \Rightarrow l = 3\cdot(1.1)^5 \Rightarrow l = 3\cdot(1.61051) \Rightarrow l = 4.83$$

j) A dealer bought a painting for $20,000 and three years later sold it for $26,620. Assuming the value of the painting increases geometrically each year, find the average rate per year that the picture is increasing.

$$a = 20000, n = 4, l = 26620$$

$$l = ar^{n-1} \Rightarrow 26620 = 20000(r)^{4-1} \Rightarrow 1.331 = r^3 \Rightarrow \sqrt[3]{1.331} = r \Rightarrow r = 1.1$$

k) The half-life of the Uranium isotope is 20.8 days, that is, one-half the given amount of Uranium 230 decomposes very 20.8 days. How much of an initial amount of 1000 grams of the isotope will be left after 208 days?

$$a = 1000, n = 11, r = \tfrac{1}{2}$$

$$l = ar^{n-1} \Rightarrow l = 1000\left(\tfrac{1}{2}\right)^{11-1} \Rightarrow l = 1000\left(\tfrac{1}{2}\right)^{10} \Rightarrow l = 1000\left(\tfrac{1}{1024}\right) \Rightarrow l = \frac{1000}{1024} = \frac{125}{128}$$

l) In 1980 the population of a small rural Saskatchewan town was 840 individuals. Fifteen years later the population of the town had dropped to 600 individuals. Calculate the rate of change in population growth.

$$a = 840, n = 16, l = 600$$

$$l = ar^{n-1} \Rightarrow 600 = 840r^{16-1} \Rightarrow \frac{600}{840} = r^{15} \Rightarrow \sqrt[15]{\frac{5}{7}} = \sqrt[15]{r^{15}} \Rightarrow \sqrt[15]{\frac{5}{7}} = r \Rightarrow$$

$r = 0.97781$ or pop is declining at a rate of 2.219 percent