Matrices

1. Definition, Basic concepts, Types
2. Addition and Subtraction of Matrices
3. Scalar Multiplication
4. Assignment and answer key
5. Matrix Multiplication
6. Assignment and answer key
7. Determinant
   - 2 x 2
   - 3 x 3 (diagonal, minors, properties)
   - summary
8. Inverse of a 2 x 2 matrix
9. Applications:
   - systems
   - Cramer's Rule
   - Problem Solving
   - Assignment and answer key
10. Practice Exam and Answer Key

Definition:

A matrix is a rectangular arrangement of elements (entries) presented between brackets [ ] or double lines || ||

Examples:

$$\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \quad \begin{bmatrix} a & c & g & i \end{bmatrix} \quad \begin{bmatrix} -5 \\ 4 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{Vmatrix} 3 & -2 & 4 \\ 5 & 2 & 7 \\ -3 & 1 & -2 \end{Vmatrix}$$

Basic Concepts:

Naming of a matrix: capital letters are used to denote a matrix

Matrix Dimension: the number of rows (horizontal) by the columns (vertical)

Given: $\begin{bmatrix} -3 & 0 & 4 \\ 6 & -2 & 5 \end{bmatrix}$

We can refer to this matrix as:

$A_{2\,x\,3}$ which translates as

matrix A having dimensions 2 by 3

Note: dimension is always given as a subscript and the variable "x" reads "by" not times.

Types of matrices:

a) row matrix - contains only one row $\begin{bmatrix} 2 & -5 & 1 \end{bmatrix}$

b) column matrix - contains only one column $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

c) zero matrix - all entries are zero $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d) transpose matrix - interchanging rows and columns

$$A = \begin{bmatrix} -2 & 5 & 1 \\ 4 & 7 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} -2 & 4 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

Addition and Subtraction of Matrices

Just as the sum or difference of two real numbers is a unique real number, the sum or difference of two matrices is a unique matrix.

$$\begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & -4 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 8 \\ -3 & -7 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} (2+(-4)) & (-1+0) & (3+8) \\ (5+(-3)) & (4+(-7)) & ((-4)+3) \end{bmatrix} =$$

$$\begin{bmatrix} -2 & -1 & 11 \\ 2 & -3 & -1 \end{bmatrix} \qquad A_{2x3} + B_{2x3} = C_{2x3}$$

Only matrices of the same dimensions can be added or subtracted. If dimensions are different the addition or subtraction is undefined.

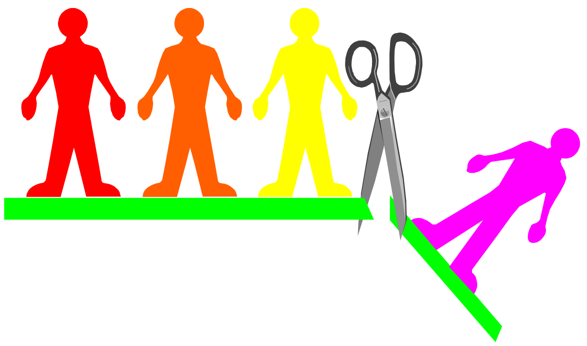
Matrix addition is both commutative and associative:

Given: $A = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$

Using the given information prove that the given statement is true.

A) Commutative

$A + B = B + A$ or $B + C = C + B$



B) Associative

$A + (B + C) = (A + B) + C$

The sum of the zero matrix O_{mxn} and any other matrix A_{mxn} is A_{mxn} , the zero matrix is the identity element for addition in the set of any matrix having dimensions *m x n*

Example: $$\begin{bmatrix} -3 & 4 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 2 & 7 \end{bmatrix}$$

The additive inverse (or negative) of the matrix A_{mxn} is the matrix - A_{mxn} whose entries are the negative of the corresponding entries in A.

$$\text{if } A = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} -4 & 1 \\ -3 & -2 \end{bmatrix},$$

$$\text{because } \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

As with real number, the concept of subtraction of matrices can be redefined to as follows:

the addition of an additive inverse

$$A_{mxn} - B_{mxn} = A_{mxn} + (-B_{mxn})$$

$$\begin{bmatrix} 2 & 4 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} -3 & 7 \\ 2 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ -2 & -4 \end{bmatrix} =$$

$$\begin{bmatrix} 5 & -7 \\ -4 & 1 \end{bmatrix}$$

Scalar Multiplication

When we work with matrices we refer to any real number as a scalar.

The product of a scalar "*s*" and a matrix A_{mxn} is the matrix $s\,A_{mxn}$ each of whose entries are "*s*" times the corresponding entry in *A*.

Example:

$$5\begin{bmatrix} -4 & 2 & 0 \\ 1 & 5 & -7 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -20 & 10 & 0 \\ 5 & 25 & -35 \\ 15 & 15 & 10 \end{bmatrix}$$

The product of a scalar and a matrix can be shown to be both commutative and associative. This proof is left up to you.

Assignment:

Given:

$$A=\begin{bmatrix}4 & -3\\ 2 & 1\end{bmatrix},\ B=\begin{bmatrix}2 & -4 & 5\end{bmatrix},\ C=\begin{bmatrix}4\\ -3\end{bmatrix},\ D=\begin{bmatrix}-3 & 1\\ 9 & 5\end{bmatrix},\ E=\begin{bmatrix}5 & 3\\ 2 & 4\end{bmatrix}$$

1. Determine the dimensions of matrices *A, B,* and *C*
2. What is the zero matrix for matrix *B*?
3. What is the transpose of matrix *B*? of matrix *E*?
4. What is the additive inverse for matrix *A*? for matrix *D*?
5. Determine the resultant matrix based on the following operations:
   a) 4 x *A* or 4*A*
   b) -7 x *C* or -7*C*
   c) (*A* + *D*) + *E*
   d) 2*A* - 3*E*
   e) *D* - (*A*+ *E*)

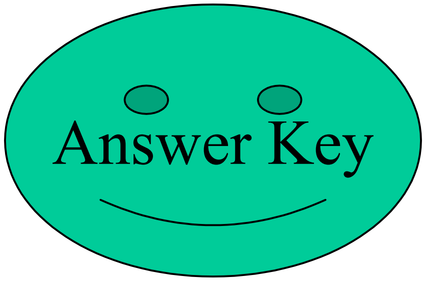
6. Solve for the variable matrix:

a)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 3\begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 8 \\ 1 & 3 \end{bmatrix}$$

b)

$$-2\begin{bmatrix} x & y \\ u & v \end{bmatrix} + 4\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} = -5\begin{bmatrix} 6 & -2 \\ 1 & 4 \end{bmatrix}$$



1. $A_{2 \times 2}$, $B_{1 \times 3}$, $C_{2 \times 1}$

2. $B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

3. $B^T = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$, $E^T = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} -4 & 3 \\ -2 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -1 \\ -9 & -5 \end{bmatrix}$

5.a) $\begin{bmatrix} 16 & -12 \\ 8 & 4 \end{bmatrix}$

5.b) $\begin{bmatrix} -28 \\ 21 \end{bmatrix}$

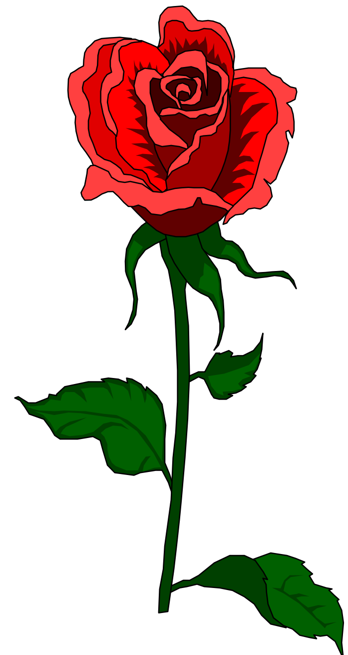
c) $\begin{bmatrix} 6 & 1 \\ 13 & 10 \end{bmatrix}$

d) $\begin{bmatrix} -7 & -15 \\ -2 & -10 \end{bmatrix}$

e) $\begin{bmatrix} -12 & 1 \\ 5 & 0 \end{bmatrix}$

6.a) $\begin{bmatrix} 0 & -8 \\ -4 & -10 \end{bmatrix}$

b) $\begin{bmatrix} -9 & 3 \\ -\frac{13}{2} & -6 \end{bmatrix}$



Matrix Multiplication

Given: $A = \begin{bmatrix} x & y & z \end{bmatrix}$ and $B = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ with the numbers *x, y,* and *z* representing the number of cassettes, CD's and DVD's sold each week at a local retail outlet while the a_1, b_1, and c_1 represent the price of each item during a regular sales week and a_2, b_2, and c_2 represent the price of each item during a special promotions week.

Problem: what were the gross receipts for the first weeks sales and what were they during the special promotion.

Note:

Gross amount on one item = number of items *x* cost per item

Gross receipts = total of gross amounts for the various items

Receipts for 1st week

Gross receipts for the first week=

cost of cassettes is xa_1, of CD's yb_1 and of DVD's zc_1 or

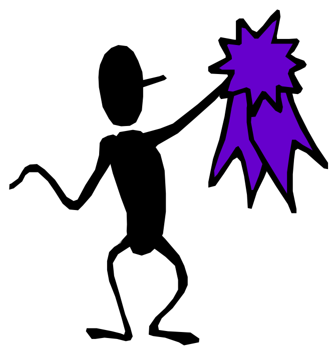
$xa_1 + yb_1 + zc_1$

Gross receipts for the special promotion=

cost of cassettes is xa_2, of CD's yb_2 and of DVD's zc_2 or

$xa_2 + yb_2 + zc_2$

The process of adding the products obtained by multiplying the elements of a row in one matrix by the corresponding elements of a column in another matrix defines the process of
matrix multiplication



It is important to note that for matrix multiplication to occur:

a) the number of elements in a row in the first matrix must match the number of elements in a column in the second matrix

b) the resulting answer will always have dimensions defined by the number of rows in the first matrix and the number of columns in the second matrix

Represent the dimensions of resulting matrix

$$A_{a \times b} * B_{c \times d} = C_{a \times d}$$

If equal, matrix multiplication can occur

General Notes:

Matrix multiplication differs from that of real numbers in that it is not, in general, commutative.

$$\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \bullet \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 26 \\ 16 & 43 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix} \bullet \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 21 & 47 \\ 14 & 32 \end{bmatrix}$$

Therefore, when it is necessary to find a product we must pay strict attention to the order in which it is written. *AB* means *left-multiplication* of *B* by *A*, and *BA* means *right-multiplication* of *B* by *A*.

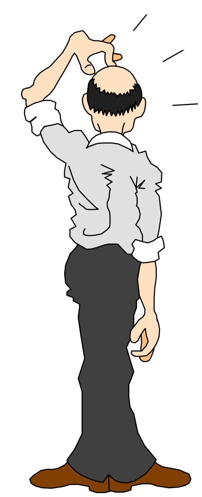
$$\text{if } A = \begin{bmatrix} 3 & -1 \\ 5 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 6 \\ 1 & 7 \end{bmatrix}, AB = \begin{bmatrix} 5 & 11 \\ 14 & 58 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 36 & 22 \\ 38 & 27 \end{bmatrix}$$

If we have a square matrix whose main diagonal (from upper left to lower right) consists of entries of "I" and all other entries are "0", we refer to this as an identity matrix and is labeled "I". The following are examples of identity matrices for 2 x 2 and 3 x 3 matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Whether we use left or right multiplication, the identity matrix times a given matrix results in the given matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -2 & 4 \end{bmatrix}$$



Given: $\begin{bmatrix} 3 & 5 \\ -6 & -10 \end{bmatrix}\begin{bmatrix} -5 & -10 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ illustrates that even though the product of two matrices equals a zero matrix **does not imply** that the first matrix must equal zero or that the second matrix must equal zero.

If we were to compare this with the real number system and were given $xy=0$, then either $x = 0$ or $y = 0$

Two valid laws are left up to you to prove: Use the given matrices and prove the following:

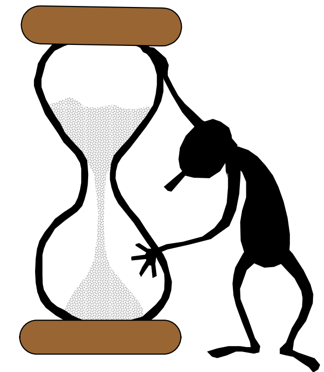
a) The associative law $(AB)C= A(BC)$

b) The distributive laws: $AB + AC = A(B + C)$ and

$$BA+CA = (B + C)A$$

$$A=\begin{bmatrix} 3 & -2 \\ 4 & 7 \end{bmatrix}, \quad B=\begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix}, \quad C=\begin{bmatrix} -4 & 3 \\ 2 & -8 \end{bmatrix}$$

Assignment:



1. $\begin{bmatrix} 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 1 & 4 \\ 3 & -2 & 1 \\ 7 & 6 & 3 \end{bmatrix} \begin{bmatrix} -2 & 4 & -4 \\ 3 & 2 & 6 \\ -1 & 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 6 & 2 \\ -3 & 9 & 4 \\ 5 & 0 & -3 \end{bmatrix} \begin{bmatrix} 5 & 4 & 2 \\ -2 & 7 & 4 \\ 3 & -4 & -5 \end{bmatrix}$

7. $\begin{bmatrix} 3 & -9 & 2 \\ 7 & 4 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -2 & -7 \\ 6 & 2 \end{bmatrix}$

Answers:

1. $\begin{bmatrix} 29 \end{bmatrix}$

2. $\begin{bmatrix} 18 & 8 \\ -5 & 43 \end{bmatrix}$

3. $\begin{bmatrix} -26 \end{bmatrix}$

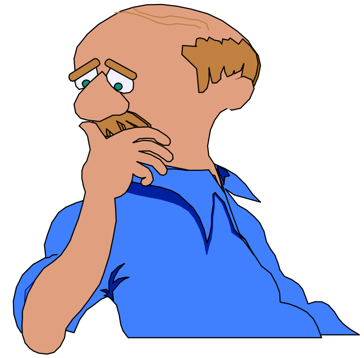
4. $\begin{bmatrix} 22 & -7 \\ -22 & 9 \end{bmatrix}$

5. $\begin{bmatrix} -9 & 26 & -18 \\ -13 & 10 & -26 \\ 1 & 46 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 14 & 50 & 22 \\ -21 & 35 & 10 \\ 34 & 8 & -5 \end{bmatrix}$

7. $\begin{bmatrix} 42 & 76 \\ 56 & 5 \end{bmatrix}$

The Determinant Function δ



The association of a real number with square matrices of any dimension (order) is called the determinant of the matrix.

Determinants can be distinguished from a matrix because they are always enclosed within a single set of vertical lines "$| \;\; |$". The entries are called elements of the determinant and the number of entries in any row or column is called the order of the determinant.

The value of the real number defined by the array is calculated by following definite rules referred to as "expanding the determinant".

To calculate the value of the determinant

A) For a 2 x 2 determinant:

Great, a formula that makes it easy

Given: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then: $\delta A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example #1:

$B = \begin{bmatrix} 7 & 2 \\ -3 & 8 \end{bmatrix}$

$\delta B = (7)(8) - (2)(-3)$

$= 56 + 6$

$= 62$

Example #2:

$\begin{vmatrix} -7 & -2 \\ 3 & -6 \end{vmatrix}$

$= (-7)(-6) - (-2)(3)$

$= 42 + 6$

$= 52$

Assignment:

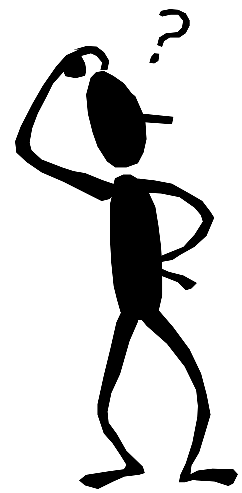
Calculate the value of each 2 x 2 determinant

1. $\begin{vmatrix} 7 & -2 \\ -5 & 8 \end{vmatrix}$

2. $\begin{vmatrix} -3 & 2 \\ 6 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 11 & -4 \\ -5 & 8 \end{vmatrix}$

4. $\begin{vmatrix} -3 & -2 \\ -8 & 4 \end{vmatrix}$



Answer Key:

1. 46

2. -33

3. 68

4. -28

B. For a 3 x 3 determinant

Procedure #1- diagonal method

goal = to display a sum of products that includes all possible arrangements of the subscripts of the letters a, b and c.

Given: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

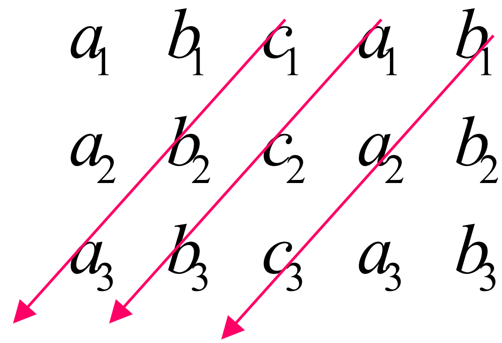
Copy the first two columns in order to the right of the 3rd column

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

Multiply each entry in first row by other two entries in the diagonal moving from left to right (these products are the first three terms of determinant)

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

Multiply each element in the row, starting from the right, by the other entries on the diagonal going right to left. The negatives of these products are the last three terms of the determinant

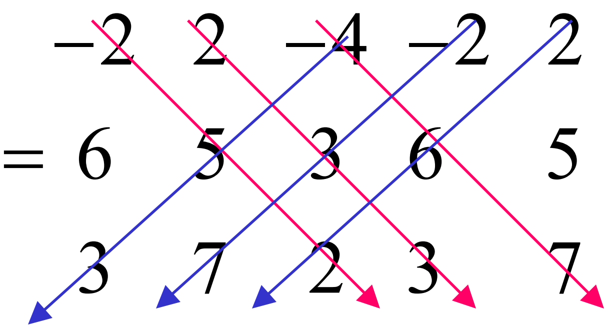


$$\delta = (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_2 b_1 c_3 + a_1 b_3 c_2 + a_3 b_2 c_1)$$

Example:

$$A = \begin{bmatrix} -2 & 2 & -4 \\ 6 & 5 & 3 \\ 3 & 7 & 2 \end{bmatrix}$$

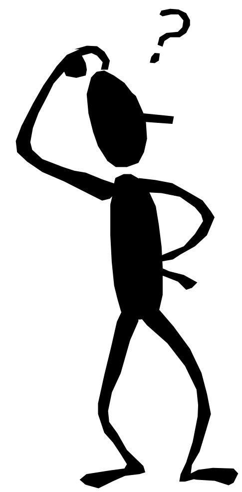
$\delta A =$



$$\delta A = ((-2)(5)(2) + (2)(3)(3) + (-4)(6)(7)) -$$
$$((-4)(5)(3) + (-2)(3)(7) + (2)(6)(2))$$
$$= (-20 + 18 - 168) - (-60 - 42 + 24)$$
$$= -170 + 78$$
$$= -92$$

Assignment:

Calculate the value of each determinant using the diagonal method



1. $$\begin{vmatrix} 4 & 6 & 2 \\ -3 & -5 & 8 \\ 1 & 4 & 3 \end{vmatrix}$$

2. $$\begin{vmatrix} -12 & 5 & 4 \\ 4 & -3 & 2 \\ 6 & 1 & 3 \end{vmatrix}$$

3. $$\begin{vmatrix} 6 & 2 & -4 \\ 7 & 4 & 5 \\ 2 & -3 & 3 \end{vmatrix}$$

4. $$\begin{vmatrix} 3 & 4 & 7 \\ -6 & -5 & 3 \\ 4 & 1 & 2 \end{vmatrix}$$

Answer Key:

1. -100

2. 220

3. 256

4. 155

Procedure #2 - Expansion By Minors:

A minor of an element in a determinant is the determinant resulting from the deletion of the row and column containing the element.

Given: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The minor of b_2 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

The minor of a_3 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

Steps:

1. Multiply each element in the chosen row or column by its minor.
2. Determine whether each element is to be assigned a "+1" or a "-1". If the sum of the number of the row and the number of the column containing the element is odd assign a "-1" and if even assign a "+1"
3. Add the resulting products

Example:

$$\begin{vmatrix} 7 & 3 & 4 \\ -2 & 6 & -1 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= (-)3\begin{vmatrix} -2 & -1 \\ 4 & 5 \end{vmatrix} + (+)6\begin{vmatrix} 7 & 4 \\ 4 & 5 \end{vmatrix} + (-)(-2)\begin{vmatrix} 7 & 4 \\ -2 & -1 \end{vmatrix}$$

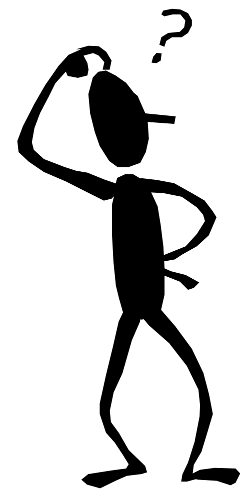
$$= -3(-10-(-4)) + 6(35-16) + 2(-7-(-8))$$

$$= -3(-6) + 6(19) + 2(1)$$

$$= 134$$

Assignment:

Determine the value of each determinant using expansion by minors.



A. About indicated row or column

1. $\begin{vmatrix} 4 & 5 & 8 \\ -6 & -3 & -1 \\ 2 & 6 & 7 \end{vmatrix}$; column 2

2. $\begin{vmatrix} -5 & 8 & 1 \\ 7 & -3 & 2 \\ 2 & 4 & 4 \end{vmatrix}$; row 3

B. About any row or column

3. $\begin{vmatrix} 9 & 2 & 3 \\ 3 & -5 & 1 \\ -4 & 1 & 6 \end{vmatrix}$

4. $\begin{vmatrix} 11 & -3 & 6 \\ 2 & 0 & 5 \\ 7 & 4 & -2 \end{vmatrix}$

Answer Key:

1.-100

2. -58

3. -374

4.-289

Procedure #3: Using properties of determinants

Property #1: You can create the negative of an original determinant by interchanging any two rows or columns.

$$\begin{vmatrix} -5 & 2 & 6 \\ 3 & 4 & 3 \\ 9 & -1 & 1 \end{vmatrix} = -221 \text{ and } \begin{vmatrix} 2 & -5 & 6 \\ 4 & 3 & 3 \\ -1 & 9 & 1 \end{vmatrix} = 221$$

Property #2: The determinant has a value of "0" if two rows or columns are identical

$$\begin{vmatrix} 3 & 5 & -7 \\ 7 & 8 & -2 \\ 7 & 8 & -2 \end{vmatrix} = 0$$

Property#3: The determinant has a value of "0" if one row or column has "0" for every element

$$\begin{vmatrix} -3 & 0 & 2 \\ 5 & 0 & 7 \\ 7 & 0 & 3 \end{vmatrix} = 0$$

Property #4: We can create a determinant equal to the original if we interchange all the rows and columns in order.

$$\begin{vmatrix} 4 & 3 & 2 \\ 1 & 7 & 5 \\ -2 & 1 & -4 \end{vmatrix} = -120 \quad \text{and} \quad \begin{vmatrix} 4 & 1 & -2 \\ 3 & 7 & 1 \\ 2 & 5 & -4 \end{vmatrix} = -120$$

Property #5: We can create a determinant "k" times the original if we multiply the elements of one row or column by a real number "k".

$$\begin{vmatrix} 4 & 1 & 2 \\ 2 & 4 & -3 \\ -6 & 3 & 1 \end{vmatrix} = 128 \quad \text{and} \quad \begin{vmatrix} 12 & 1 & 2 \\ 6 & 4 & -3 \\ -18 & 3 & 1 \end{vmatrix} = 384$$

Multiply 1st column by "3"

And now for the most important property

Property #6: If each element in any row or column is multiplied by a real number "k" and the resulting products are added to the corresponding elements of another row or column, the resulting determinant equals the original.

$$\begin{vmatrix} 4 & 3 & 8 \\ 2 & 4 & 2 \\ -5 & 7 & -1 \end{vmatrix} = 176$$

Multiply column 3 by 4 and add to column 1

$$\begin{vmatrix} 4+4(8) & 3 & 8 \\ 2+4(2) & 4 & 2 \\ -5+4(-1) & 7 & -1 \end{vmatrix} = \begin{vmatrix} 36 & 3 & 8 \\ 10 & 4 & 2 \\ -9 & 7 & -1 \end{vmatrix} = 176$$

Why is this so important??

This allows the manipulation of the elements of the determinant with the goal of maximizing the number of elements equal to zero which in turn allows expansion by minors to focus on only one minor and a 2x2 determinant.

Example #1:

$$\begin{vmatrix} 2 & 4 & 5 \\ -1 & 7 & 4 \\ 3 & -2 & -7 \end{vmatrix}$$

Multiply 2nd row by '2" and add to 1st row

$$\begin{vmatrix} 2+2(-1) & 4+2(7) & 5+2(4) \\ -1 & 7 & 4 \\ 3 & -2 & -7 \end{vmatrix} = \begin{vmatrix} 0 & 18 & 13 \\ -1 & 7 & 4 \\ 3 & -2 & -7 \end{vmatrix}$$

Multiply 2nd row by '3" and add to 3rd row

$$\begin{vmatrix} 0 & 18 & 13 \\ -1 & 7 & 4 \\ 3+3(-1) & -2+3(7) & -7+3(4) \end{vmatrix} = \begin{vmatrix} 0 & 18 & 13 \\ -1 & 7 & 4 \\ 0 & 19 & 5 \end{vmatrix}$$

Expand by minors

$$-(-1)\begin{vmatrix} 18 & 13 \\ 19 & 5 \end{vmatrix} = 1(18(5) - 13(19)) = -157$$

Example #2:

x-1

$$\begin{vmatrix} 5 & 6 & -4 \\ 3 & 4 & 7 \\ -2 & 5 & -11 \end{vmatrix} = \begin{vmatrix} 5 & 6+-1(5) & -4 \\ 3 & 4+-1(3) & 7 \\ -2 & 5+-1(-2) & -11 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 1 & -4 \\ 3 & 1 & 7 \\ -2 & 7 & -11 \end{vmatrix} = \begin{vmatrix} 5+-1(3) & 1+-1(1) & -4+-1(7) \\ 3 & 1 & 7 \\ -2+-7(3) & 7+-7(1) & -11+-7(7) \end{vmatrix}$$

x-1

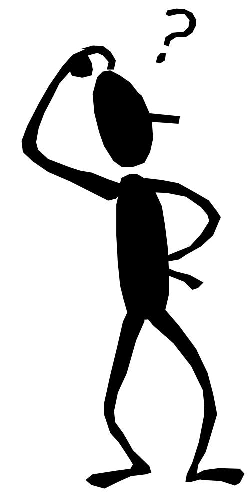
x-7

$$= \begin{vmatrix} 2 & 0 & -11 \\ 3 & 1 & 7 \\ -23 & 0 & -60 \end{vmatrix} = 1\begin{vmatrix} 2 & -11 \\ -23 & -60 \end{vmatrix}$$

$$= (2(-60) - (-11)(-23)) = -373$$

Assignment

Calculate the value of the determinant using the properties of determinants and creating zeros in rows and columns



1 $\begin{vmatrix} 9 & 6 & 7 \\ -3 & -2 & 5 \\ 4 & 8 & -3 \end{vmatrix}$

2 $\begin{vmatrix} 4 & 6 & -2 \\ -2 & 3 & 4 \\ 1 & 7 & 8 \end{vmatrix}$

3. $\begin{vmatrix} 11 & 7 & 8 \\ 13 & 4 & 5 \\ 15 & -3 & 11 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 6 & 3 & -2 \\ 8 & -1 & 1 & 5 \\ -2 & 5 & 3 & 4 \\ 3 & 2 & 7 & 2 \end{vmatrix}$

Answer Key:

1.-352

2. 138

3. -619

4. 2405

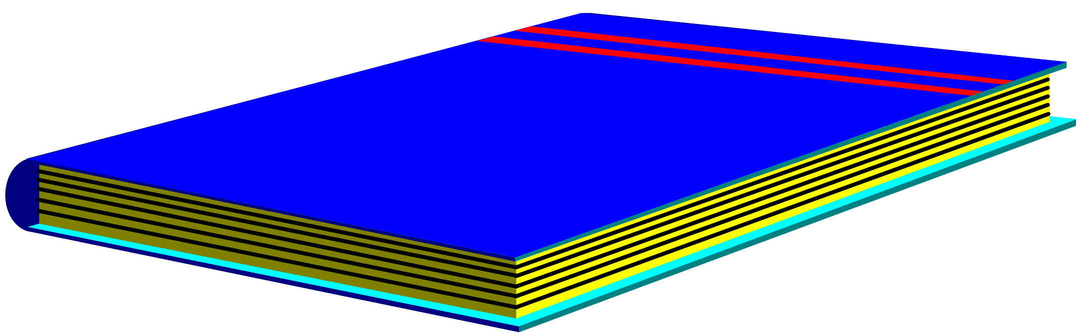
Summary about determinants:

1. The formula ad-bc works with an y 2 x 2 determinant

2. The diagonal method only works with 3 x 3 determinants

3. Expansion by minors will work with any square matrix with dimensions three or greater.

4. The creation of zeros (row and column operations) provides the easiest approach to determine the determinant and minimize the number of calculation errors.



The Inverse of a 2 x 2 Matrix

Remember: $$\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 8-7 & -28+28 \\ 2-2 & -7+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

results in the identity matrix. When the product of two real numbers is the multiplicative identity I, the two numbers are called multiplicative inverses. Similarly, any two matrices A and B such that $AB = BA = I$, are called inverses. To identity the inverse of matrix A, it is customary to substitute A^{-1} for B.

How do we calculate the inverse??

Sure hope we have a formula!!

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1}=\begin{bmatrix} u & v \\ x & y \end{bmatrix}$ such that $AA^{-1}=I$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & v \\ x & y \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} au+bx & av+by \\ cu+dx & cv+dy \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the above statement to be true:

$au + bx = 1 \qquad av + by = 0$

$cu + dx = 0 \qquad cv + dy = 1$

If $ad - bc \neq 0$, you can solve these pairs of equations (sorry, this great exercise is left up to you) for u and x, and for v and y, respectively, and find:

$$u = \frac{d}{ad - bc}, \quad v = \frac{-b}{ad - bc}$$

$$x = \frac{-c}{ad - bc}, \quad y = \frac{a}{ad - bc}$$

Since each denominator is δA, we can write A^{-1} as:

$$A^{-1} = \frac{1}{\delta A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In Summary: To get A^{-1}

a) interchange the *a* and *d*

b) change the signs on *b* and *c*

c) multiply the resulting matrix by the determinant of *A*.

Examples:

a) Determine the inverse matrix for the given matrix:

$$A = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}; \quad A^{-1} = \frac{1}{(4)(5)-(2)(-3)} \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{26} & \frac{-2}{26} \\ \frac{3}{26} & \frac{4}{26} \end{bmatrix}$$

b) Solve for matrix *A*

$$\begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} A = \begin{bmatrix} 2 & 2 \\ -3 & 5 \end{bmatrix}$$

$$\frac{1}{-6-4} \begin{bmatrix} -2 & -1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} A = \frac{1}{-6-4} \begin{bmatrix} -2 & -1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A = \frac{-1}{10} \begin{bmatrix} 1 & -9 \\ -17 & 7 \end{bmatrix} = \begin{bmatrix} \frac{-1}{10} & \frac{9}{10} \\ \frac{17}{10} & \frac{-7}{10} \end{bmatrix}$$

Assignment:

A. Determine the inverse of the following matrices:

1. $\begin{bmatrix} 5 & 3 \\ -8 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} -11 & 8 \\ 9 & -2 \end{bmatrix}$

B. Solve for the matrix A.

1. $\begin{bmatrix} -3 & 2 \\ 6 & 5 \end{bmatrix} A = \begin{bmatrix} 8 & 4 \\ -2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 9 & 4 \\ -2 & 5 \end{bmatrix} A = \begin{bmatrix} -7 & 1 \\ 3 & 2 \end{bmatrix}$

Answer Key:

A. 1. $\begin{bmatrix} \frac{4}{44} & \frac{-3}{44} \\ \frac{8}{44} & \frac{5}{44} \end{bmatrix}$ 2. $\begin{bmatrix} \frac{-2}{-50} & \frac{-8}{-50} \\ \frac{-9}{-50} & \frac{-11}{-50} \end{bmatrix}$

B. 1. $\begin{bmatrix} \frac{-44}{27} & \frac{-14}{27} \\ \frac{14}{9} & \frac{11}{9} \end{bmatrix}$ 2. $\begin{bmatrix} \frac{-47}{53} & \frac{-3}{53} \\ \frac{13}{53} & \frac{20}{53} \end{bmatrix}$

Applications of Matrices:

a) Solving Systems of Equations:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix}$$

⇒ Reason - matrix multiplication

If we are given: $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ it is possible to write an equivalent matrix equation:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Which can be translated into:

coefficient matrix X variable matrix = constant matrix

To solve:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Remember the use of the multiplicative inverse

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\delta\ A}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{b_2c_1 + (-b_1)c_2}{\delta\ a} \\ \dfrac{(-a_2)c_1 + a_1c_2}{\delta\ A} \end{bmatrix}$$

Solution Set:

$$\left\{\left(\frac{b_2c_1 + (-b_1)c_2}{\delta\ a}, \frac{(-a_2)c_1 + a_1c_2}{\delta\ A}\right)\right\}$$

Example #1

Find the solution set for the given system:

$$3x - 5y = 7$$

$$4x + 3y = -4$$

$$\begin{bmatrix} 3 & -5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$ → Matrix equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{29} \begin{bmatrix} 3 & 5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$ → Multiplicative inverse

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{29} \\ \frac{-40}{29} \end{bmatrix}$$ → Solution Set {(1/29, -40/29)}

Example #2

Find the solution set for the following problem. Use matrices.

The measure of one of two complementary angles is 2 degrees less than three times that of the other. How large are the angles?

Let x = measure of larger angle in degrees
y = measure of smaller angle in degrees

Open sentences: $x + y = 90$
$x = 3y - 2$

To use matrices it is required that:

1. Both variables must be on the left side of the equal sign and the constant on the right side
2. That the order of variables be maintained in both questions (use order based on position in alphabet)

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 90 \\ -2 \end{bmatrix}$$

⇒ Matrix equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 90 \\ -2 \end{bmatrix}$$

⇒ Multiplicative Inverse

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 67 \\ 23 \end{bmatrix}$$

Solution Set: $\{(67, 23)\}$

B. Cramer's Rule

What is this? A matrix procedure that can be used to solve systems of equations - especially equations that have three or more variables.

To understand the rule let us first examine the procedure as it would apply to a system of equations of two variables

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

Let $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $xD = \begin{vmatrix} a_1 x & b_1 \\ a_2 x & b_2 \end{vmatrix}$

Coefficient matrix

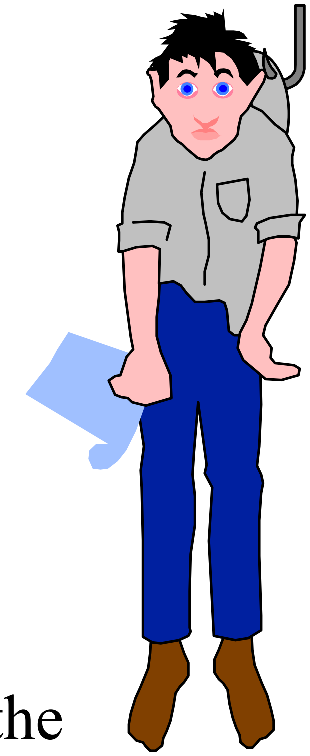
Property #5

Apply Property #6

$$xD = \begin{vmatrix} a_1 x + b_1 y & b_1 \\ a_2 x + b_2 y & b_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

If $D \neq 0,$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}$$



In summary:

1. Create a coefficient matrix D
2. Create a D_x matrix by replacing the x coefficients in the coefficient matrix with the constant values
3. The ratio D_x/D defines the value for the variable x.
4. Similarly, create matrix D_Y by replacing the y coefficients in the coefficient matrix with the constant values
5. The ratio D_y/D defines the value for the variable y.
6. The replacement of coefficients with constants and the creation of the ratio is repeated equal to the number of different variables

Example #1

Determine the solution set for the following system of equations using Cramer's Rule

5x - 2y = 7
3x + 7y = 2

$$D = \begin{vmatrix} 5 & -2 \\ 3 & 7 \end{vmatrix} = (5)(7) - (-2)(3) = 41$$

$$D_x = \begin{vmatrix} 7 & -2 \\ 2 & 7 \end{vmatrix} = (7)(7) - (-2)(2) = 53$$

$$D_y = \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} = (5)(2) - (7)(3) = -11$$

$$x = \frac{D_x}{D} = \frac{53}{41}, y = \frac{D_y}{D} = \frac{-11}{41}$$

Solution Set: {(53/41,-11/41)}

Example #2

Determine the solution set for the following system:
(Remember to calculate the value of the various determinants you can use the diagonal method or row or column operations which result in zeros)

4x - 2y + 5z = -2
-3x + 4y - 3z = 6
x + 2y - 4z = 5

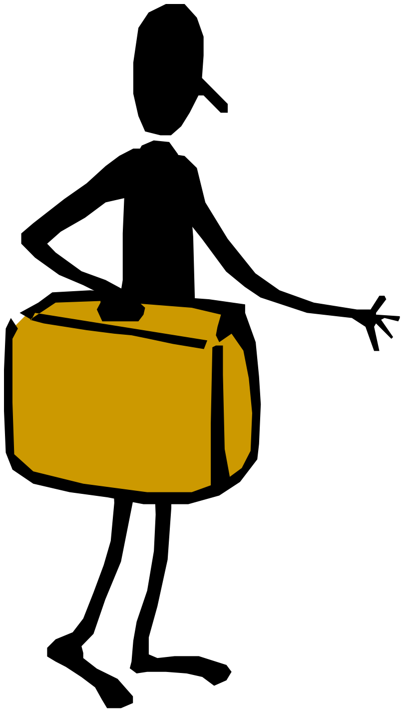
$$D = \begin{vmatrix} 4 & -2 & 5 \\ -3 & 4 & -3 \\ 1 & 2 & -4 \end{vmatrix} = -60$$

$$D_x = \begin{vmatrix} -2 & -2 & 5 \\ 6 & 4 & -3 \\ 5 & 2 & -4 \end{vmatrix} = -38$$

Continues on next slide

$$D_y = \begin{vmatrix} 4 & -2 & 5 \\ -3 & 6 & -3 \\ 1 & 5 & -4 \end{vmatrix} = -111$$

$$D_z = \begin{vmatrix} 4 & -2 & -2 \\ -3 & 4 & 6 \\ 1 & 2 & 5 \end{vmatrix} = 10$$



$$\frac{D_x}{D} = \frac{-38}{-60}, \frac{D_y}{D} = \frac{-111}{-60}, \frac{D_z}{D} = \frac{10}{-60}$$

Solution Set: {(19/30, 111/60, -1/6)}

C. Problem Solving

We wish to calculate the maximum height a rocket will reach, when will it reach this height, and when it will hit the ground?

A tracking station was able to give the altitude of the rocket at various horizontal distances from the launch site. The collected data was as follows: altitude of 38 kilometers at a distance of 5 kms, an altitude of 72 kms at 10 kms, and an altitude of 102 kms at 15 kms.

We will assume that the object in free flight will have a path that approximates a parabola defined by equation $f(x) = ax^2 + bx + c$

To use the equation we will have to determine the values of *a, b* and *c* by substituting the coordinates of the positions we know we will create three equations.

$$f(5) = a(5)^2 + b(5) + c = 38 \text{ or } 25a + 5b + 1c = 38$$

$$f(10) = a(10)^2 + b(10) + c = 72 \text{ or } 100a + 10b + 1c = 72$$

$$f(15) = a(15)^2 + b(15) + c = 102 \text{ or } 225a + 15b + 1c = 102$$

To determine the values of *a, b* and *c* it is necessary to create the D, D_a, D_b and D_c determinants

$$D = \begin{vmatrix} 25 & 5 & 1 \\ 100 & 10 & 1 \\ 225 & 15 & 1 \end{vmatrix}, D_a = \begin{vmatrix} 38 & 5 & 1 \\ 72 & 10 & 1 \\ 102 & 15 & 1 \end{vmatrix}$$

$$D_b = \begin{vmatrix} 25 & 38 & 1 \\ 100 & 72 & 1 \\ 225 & 102 & 1 \end{vmatrix}, D_c = \begin{vmatrix} 25 & 5 & 38 \\ 100 & 10 & 72 \\ 225 & 15 & 102 \end{vmatrix}$$

$$\frac{D_a}{D} = -0.08$$

$$\frac{D_b}{D} = 8$$

$$\frac{D_c}{D} = 0$$

The equation of the path of the object is: $f(x) = -0.08x^2 + 8x$

A) to determine the horizontal distance at which the rocket reaches the maximum height

$$x = -\frac{b}{2a} = -\frac{8}{2(-0.08)} = 50$$

maximum height at 50 kms form launch

B) to calculate the maximum height

$$F(50) = -0.08(50)^2 + 8(50) = -200 + 400 = 200$$

max height is 200 kilometers

C) when the rocket hits the ground (the altitude of the rocket will be 0)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 - 4(-0.08)(0)}}{2(-0.08)} =$$

$$\frac{-8 \pm 8}{-0.16} = 0 \text{ and } 100$$

0 = launch position

100 = point of impact

Assignment:

1. Solve the following system of equations using the idea of multiplicative inverses.

a) $5x - 7y = 2$
$-4x + 3y = -8$

b) $9x - 5y = 13$
$2x + 7y = -3$

2. Solve the following systems of equations using Cramer's Rule

a) $6x - 7y = -5$
$2x - 3y = 2$

b) $-4x + 3y + z = 7$
$5x + y - 3x = -2$
$2x + 5y - 2z = 4$

3. A rocket is tracked at three distinct points in flight. The position is recorded as pairs of the form (distance from launch pad, altitude). Find maximum height and distance to impact from launch site.
(2, 18), (4, 32), and (6, 42)

1. a) $\{(\frac{50}{13}, \frac{32}{13})\}$

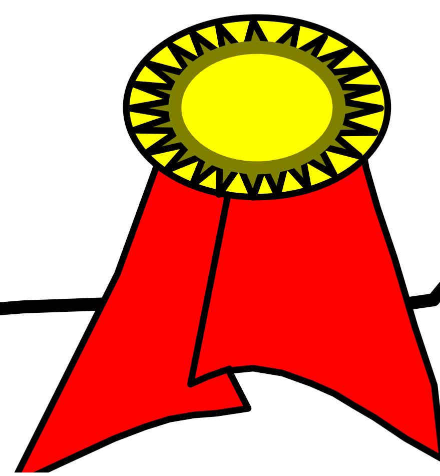
b) $\{(\frac{76}{73}, -\frac{53}{73})\}$

2. a) $\{(\frac{-29}{4}, \frac{-11}{2})\}$

b) $\{(\frac{29}{-17}, \frac{-12}{-17}, \frac{33}{-17})\}$

3. Maximum height = 50

distance to impact = 20



Practice exam:

1 Given :

$$A = \begin{bmatrix} -3 & 5 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 8 & 3 \\ -2 & -4 \end{bmatrix}, C = \begin{bmatrix} -5 & -7 \\ 2 & 2 \end{bmatrix}$$

a) A + B – C

b) 3A + 2B

c) C^T

d) A*B + C

e) B^{-1}

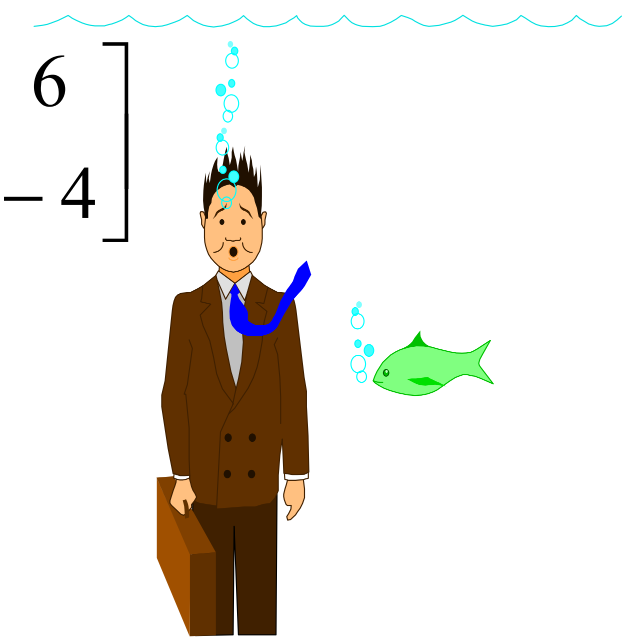
f) δ A

2..Solve:

a) $$3\begin{bmatrix} 4 & 1 \\ -2 & 7 \end{bmatrix} + 5\begin{bmatrix} a & c \\ b & d \end{bmatrix} = -4\begin{bmatrix} -2 & 6 \\ 5 & -4 \end{bmatrix}$$

b) 5x + 2y = 9

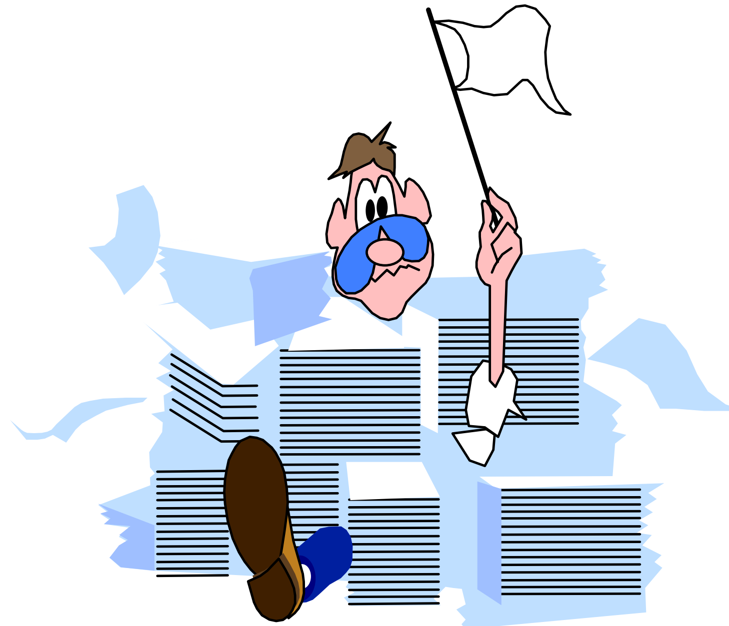
-4x + 7y= -4



c) $4x + 2y - 3z = -6$

$-2x + y + 4z = 2$

$5x + 7y - 4z = -3$



3..Multiply the following:

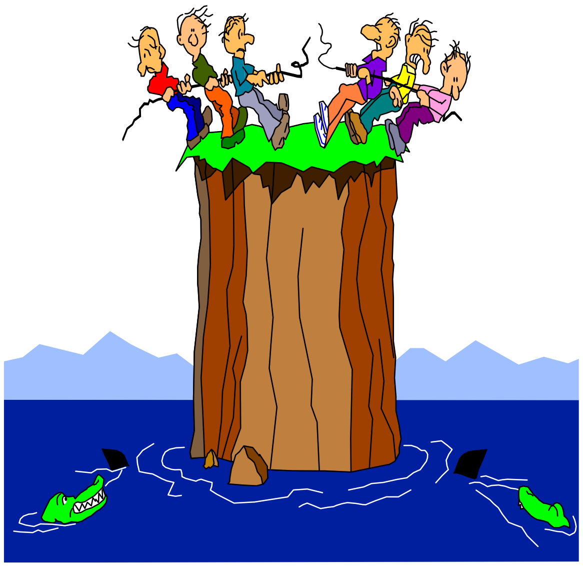
a) $$\begin{bmatrix} 4 & 1 & -4 \\ -2 & -7 & 3 \end{bmatrix}\begin{bmatrix} 8 & 5 \\ -2 & -1 \\ 3 & 6 \end{bmatrix}$$

b) $$\begin{bmatrix} 5 & 0 & 2 \\ -2 & 7 & -3 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 & 5 \\ -2 & -4 & 3 \\ 3 & 2 & -4 \end{bmatrix}$$

4.Determine the value of the determinant using the indicated method:

a) diagonal

$$\begin{vmatrix} 4 & 3 & 8 \\ -6 & -5 & 4 \\ 2 & -2 & -1 \end{vmatrix}$$



b) expansion by minors

$$\begin{vmatrix} 7 & -2 & 8 \\ -3 & 1 & 4 \\ 2 & 4 & -5 \end{vmatrix}$$

c) properties of determinants

$$\begin{vmatrix} 12 & -4 & -7 \\ -8 & 6 & 11 \\ 5 & 13 & -8 \end{vmatrix}$$

Answers:

1. a) $\begin{bmatrix} 10 & 15 \\ 0 & -4 \end{bmatrix}$ b) $\begin{bmatrix} 7 & 21 \\ 8 & -2 \end{bmatrix}$ c) $\begin{bmatrix} -5 & 2 \\ -7 & 2 \end{bmatrix}$ d) $\begin{bmatrix} -39 & -36 \\ 30 & 6 \end{bmatrix}$

e) $\begin{bmatrix} 4/26 & 3/26 \\ -2/26 & -8/26 \end{bmatrix}$ f) -26

2. A) $\begin{bmatrix} -4/5 & -27/5 \\ -14/5 & -5/5 \end{bmatrix}$ b) $\{(71/43, 16/43)\}$ c) $\{(133/-47, -44/-47, 54/-47)\}$

3. a) $\begin{bmatrix} 18 & -5 \\ 7 & 15 \end{bmatrix}$ b) $\begin{bmatrix} 31 & 9 & 17 \\ -33 & -36 & 23 \\ 14 & 15 & -8 \end{bmatrix}$

4. a) 234 b) -245 c) -1318