Simplifying Radicals Notes

Often when we have a radical expression, we need to simplify it. A radical is in simplified form if it meets 3 criteria:

- There are no perfect n^{th}-factors inside the radical
- There are no fractions inside a radical
- There are no radical signs in the denominator of a fraction

In this section, we will deal only with the first criterion – that there be no perfect nth factors inside the radical sign.

To simplify we will be using the definition of fractional powers:

Definition of Fractional Exponents

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

We have worked with the first part of this definition before, but we will use the last part in simplifying. Study the following examples:

Example. Use the definition of fractional exponents to rewrite the radical expression in exponential form.

1. $\sqrt[3]{x^5}$

   By our definition, we can rewrite this as: $\sqrt[3]{x^5} = x^{5/3}$

2. $\sqrt{y^7}$

   When we have a square root, the index is understood to be 2. We can write in the index and then proceed as above:

   $$\sqrt{y^7} = \sqrt[2]{y^7} = y^{7/2}$$

Sometimes we can simplify our result after writing in exponential form:

3. $\sqrt[3]{x^3}$

   Applying the definition we get $\sqrt[3]{x^3} = x^{3/3}$. If we simplify the exponent, $3/3 = 1$, and so the problem simplifies to $x^{3/3} = x^1 = x$

4. $\sqrt[4]{y^8}$

   Applying the definition and simplifying we get $\sqrt[4]{y^8} = y^{8/4} = y^2$

5. $\sqrt{z^{10}}$

   Again, the index of the square root is 2 and so $\sqrt{z^{10}} = z^{10/2} = z^5$

In addition to the definition of fractional exponents, we will also use the following:

Rule of Radicals 1: $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

This rule can either be used from left to right or from right to left.

Simplifying Square Root Expressions

In order to simplify a square root, we need to make sure that there are no perfect square factors inside the radical sign. If there are, then we need to simplify. To help us, it is useful to have a list of the perfect squares.

Perfect Squares from 1 to 100:
1   4   9   16   25   36   49   64   81   100

Examples: Simplify the following radicals:
1. $\sqrt{18}$
    To simplify, we first look for a factor of 18 which is a perfect square. Looking at our list of perfect squares, we see that 9 is a factor of 18 since $18 = 9 \cdot 2$. Study the steps below:
    $$\sqrt{18} = \sqrt{9 \cdot 2}$$
    $$= \sqrt{9} \cdot \sqrt{2} \quad \text{by Radical Rule 1}$$
    $$= 3 \cdot \sqrt{2} \quad \text{since } \sqrt{9} = 3$$
    $$= 3\sqrt{2}$$

2. $\sqrt{x^{13}}$
    To simplify this, we look for a "perfect square factor" of x^{13}. This will be the largest power of x which is less than 13, or 12. We replace x^{13} by $x^{12} \cdot x^1$.
    $$\sqrt{x^{13}} = \sqrt{x^{12} \cdot x^1}$$
    $$= \sqrt{x^{12}} \cdot \sqrt{x}$$
    We can now simplify $\sqrt{x^{12}}$ using the definition of fractional exponents: $\sqrt{x^{12}} = x^{12/2} = x^6$.
    This brings the first part out of the radical sign. The rest of the problem is:
    $$\sqrt{x^{12}} \cdot \sqrt{x} = x^6 \cdot \sqrt{x} = x^6 \sqrt{x}$$

3. $\sqrt{8a^4 b^9}$
    To simplify this, we look for perfect square factors as in examples 1 and 2. 8 can be written as $4 \cdot 2$. a^4 is a perfect square. b^9 can be rewritten as $b^8 \cdot b$.
    $$\sqrt{8a^4 b^9} = \sqrt{4 \cdot 2 \cdot a^4 \cdot b^8 \cdot b}$$
    $$= \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{a^4} \cdot \sqrt{b^8} \cdot \sqrt{b}$$
    We simplify $\sqrt{a^4} = a^{4/2} = a^2$ and $\sqrt{b^8} = b^{8/2} = b^4$
    $$\sqrt{4} \cdot \sqrt{2} \cdot \sqrt{a^4} \cdot \sqrt{b^8} \cdot \sqrt{b} =$$
    $$2 \cdot \sqrt{2} \cdot a^2 \cdot b^4 \cdot \sqrt{b} =$$
    $$2a^2 b^4 \sqrt{2} \cdot \sqrt{b} =$$
    $$2a^2 b^4 \sqrt{2b}$$
    In the last step, the square roots were combined using Radical Rule 1 from right to left.

Once we have learned to simplify radicals, we can use the technique to simplify radicals prior to adding or subtracting them. Study the following example:

Example: Simplify the radical expressions first and then add or subtract.

1. $\sqrt{25y} - \sqrt{81y}$

We begin by simplying each radical:

$$\sqrt{25y} - \sqrt{81y} = \sqrt{25} \cdot \sqrt{y} - \sqrt{81} \cdot \sqrt{y}$$
$$= 5\sqrt{y} - 9\sqrt{y}$$
$$= -4\sqrt{y}$$

In the last step, we note that we have like radicals and so we can combine them.

2. $\sqrt{18x^5} + x^2 \sqrt{50x}$

Again, we begin by simplifying each radical and then combining like terms:

$$\sqrt{18x^5} + x^2 \sqrt{50x} =$$
$$\sqrt{9 \cdot 2 \cdot x^4 \cdot x} + x^2 \sqrt{25 \cdot 2 \cdot x} =$$
$$\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{x^4} \cdot \sqrt{x} + x^2 \cdot \sqrt{25} \cdot \sqrt{2} \cdot \sqrt{x} =$$
$$3 \cdot \sqrt{2} \cdot x^2 \cdot \sqrt{x} + x^2 \cdot 5 \cdot \sqrt{2} \cdot \sqrt{x} =$$
$$3x^2 \sqrt{2x} + 5x^2 \sqrt{2x} =$$
$$8x^2 \sqrt{2x}$$

Simplifying Cube Roots

We can go through a similar process to simplify cube roots. This time we want to look for perfect cube factors. The list of the first few perfect cubes is:

Perfect Cubes: 1   8   27   64   125   216

Note what happens when we simplify exponents:

Example. Simplify $\sqrt[3]{x^{15}}$

Using the definition of fractional exponents, we get $\sqrt[3]{x^{15}} = x^{15/3} = x^5$.

We see that in order for the expression to simplify out of the radical, the power must be a multiple of 3.

Example: Simplify:

1. $\sqrt[3]{40x^{17}}$

We begin by looking for a perfect cube which is a factor of 40. This is $8 \cdot 5$. The x^{17} must be rewritten as $x^{15} \cdot x^2$, since 15 is a multiple of 3. The simplification proceeds as follows:

$$\sqrt[3]{40x^{17}} = \sqrt[3]{8 \cdot 5 \cdot x^{15} \cdot x^2}$$
$$= \sqrt[3]{8} \cdot \sqrt[3]{5} \cdot \sqrt[3]{x^{15}} \cdot \sqrt[3]{x^2}$$
$$= 2 \cdot \sqrt[3]{5} \cdot x^{15/3} \cdot \sqrt[3]{x^2}$$
$$= 2 \cdot \sqrt[3]{5} \cdot x^5 \cdot \sqrt[3]{x^2} = 2x^5 \sqrt[3]{5x^2}$$