

# Laboratory # 2

## An Introduction to Limits

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### Teaches:

- limits
  - one-sided limits
  - limits involving infinity – vertical and horizontal asymptotes
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1. In Calculus, we often deal with problems in which the answer cannot be obtained directly but must be approached by a series of approximations. Consider the function  $f(x) = \frac{\sin x}{x}$ ;  $f(x)$  cannot be evaluated at  $x = 0$ ; however,  $f(x)$  can be evaluated at values of  $x$  as close to 0 as we might want.

Click on Derive's Algebra Window to make it active, then choose **Declare**  $\gg$  **Algebra State**  $\gg$  **Simplification**. Change the Precision Mode to Approximate with 11 digits, then click OK. This will insure that when you simplify an expression, Derive will approximate it with 11 digits of accuracy. Author   $\frac{\sin(x)}{x}$ . Then click on Derive's graphics window, and Plot . Switch to Trace Mode (use  to toggle in and out of Trace Mode). If you use the left and right arrow keys to move the cross along the graph, you will notice that the graph appears to pass through the point (0,1) (it doesn't really since  $\frac{\sin x}{x}$  is not defined at 0 so the graph really has a tiny hole at (0,1); in fact, Derive will "Beep" as you move the cross through this point to warn you that  $f(x)$  is actually undefined at  $x = 0$ ).

2. To evaluate  $\frac{\sin x}{x}$  at, say, 1, make sure this expression is highlighted, then click the SUB button .

With the variable  $x$  highlighted, type 1 in the substitution box, and then click Simplify. Answer

Question #1.

If  $F(x)$  “gets close to” some number  $L$  as  $x$  “gets close to” a number  $a$ , we write

$$\lim_{x \rightarrow a} F(x) = L.$$

We read this as “The limit of  $F(x)$  as  $x$  approaches  $a$  is  $L$ ”. If  $F(x)$  does not approach any particular number as  $x$  gets close to  $a$ , we say the limit does not exist and write DNE.

Your calculations in Worksheet Question #1 should suggest that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

3. Answer Questions #2 and #3.

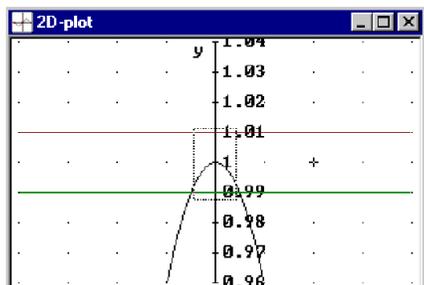
4. Let us explore the concept of a limit graphically. You should still have the plot of  $y = \frac{\sin x}{x}$  in your graphics window. Our conjecture is that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ . To obtain a better sense of this graphically, do the following:

- (a) Suppose “ $y$  close to 1” means that  $0.99 < y < 1.01$ . Author and Plot the following two expressions:

$$y = .99$$

$$y = 1.01$$

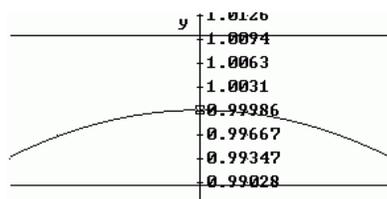
- (b) Then “ $y$  close to 1” means that  $y$  is between these two horizontal lines. Choose **Set**  $\gg$  **Scale** to change your  $y$  (vertical) scale to .01. (This will “zoom you in” on the  $y$  axis. Change your viewing window so that both horizontal lines are included, and only the part(s) of the graph that lie between these two horizontal lines are included in your viewing window. You can do this using the “Set Range with Box”  button. When you click on it, The mouse pointer will change into a large cross. Move the mouse cross to the desired upper left corner of the viewing window, then hold down the left mouse button and move the mouse cross to the desired lower right corner of the plot range:



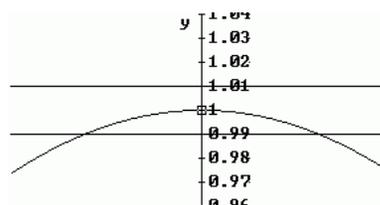
Release the mouse button. The coordinates of the new viewing window will be displayed at which point you can accept them (see Figure (a)). Your viewing window has the following properties:

- i. It contains the point  $(0, 1)$ .
- ii. For any  $x$  in the viewing window (except  $x = 0$ ),  $y = f(x)$  lies between the two horizontal lines  $y = 0.99$  and  $y = 1.01$ .

The fact that, no matter how close together your two horizontal lines  $y = 1 - \epsilon$  and  $y = 1 + \epsilon$  are, you can find a viewing window like (a) which satisfies these two properties, is what we mean by saying that the limit of  $\frac{\sin x}{x}$  as  $x$  approaches 0 is 1. Now zoom out once or twice  so that you can see where the graph “leaves” the interval  $0.99 < y < 1.01$  (see Figure (b)).



(a)



(b)

We are now ready to state the definition of a limit. As you read it, keep in mind the previous example where  $f(x) = \frac{\sin x}{x}$ ,  $a = 0$ ,  $L = 1$ , and  $\epsilon = .01$ :

**Definition 2.1** We say  $\lim_{x \rightarrow a} f(x) = L$  if for any 2 horizontal lines  $y = L + \epsilon$  and  $y = L - \epsilon$ , we can find a viewing window centered at  $(a, L)$  such that for all  $x$ 's in our viewing window (except possibly  $x = a$ ) the graph of  $f(x)$  lies between the two horizontal lines.

In trace mode, you can use your graph to find the largest interval  $I$  on which the graph lies between the two horizontal lines  $y = 0.99$  and  $y = 1.01$ . Verify that this interval is approximately  $-0.24 < x < 0.24$ .

In other words, if  $-0.24 < x < 0.24$ , then  $0.99 < y = \frac{\sin x}{x} < 1.01$ .

Answer Questions #4 and #5.

5. Question #6 reproduces the second table that you filled in in Question #3. This table suggests that

$\lim_{x \rightarrow 1} \frac{x^3 - 1}{|x - 1|}$  does not exist, since there is no unique number that  $\frac{x^3 - 1}{|x - 1|}$  approaches as  $x$  gets close to 1. However if we restrict ourselves to values of  $x$  that are close to but **greater than** 1, then we see that  $\frac{x^3 - 1}{|x - 1|}$  approaches 3; similarly, if we restrict ourselves to values of  $x$  that are close to but **less than** 1, then we see that  $\frac{x^3 - 1}{|x - 1|}$  approaches  $-3$ . We call 3 the **right limit** of  $\frac{x^3 - 1}{|x - 1|}$  as  $x$  approaches 1; we call  $-3$  the **left limit** of  $\frac{x^3 - 1}{|x - 1|}$  as  $x$  approaches 1. We write:

$$\lim_{x \rightarrow 1^+} \frac{x^3 - 1}{|x - 1|} = 3 \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{x^3 - 1}{|x - 1|} = -3$$

Reset your graphics window by deleting all your graphs , setting your scales both to 1 , and centering the window at the origin . Then Author and Plot  $\frac{x^3 - 1}{|x - 1|}$ . Use Trace Mode to explore the behavior of  $y$  as  $x$  approaches 3, first from the right, then from the left. Notice how the  $y$  values for the part of the graph to the right of  $x = 1$  are close to 3 when  $x$  is close to 1; similarly, notice how the  $y$  values for the part of the graph to the left of  $x = 1$  are close to  $-3$  when  $x$  is close to 1. Answer Questions #6 and #7.

**Definition 2.2** If  $F(x)$  increases without bound as  $x$  gets closer and closer to a number  $a$ , where  $x > a$ , we say that **the limit of  $F(x)$  as  $x$  approaches  $a$  from the right is infinity** and we write

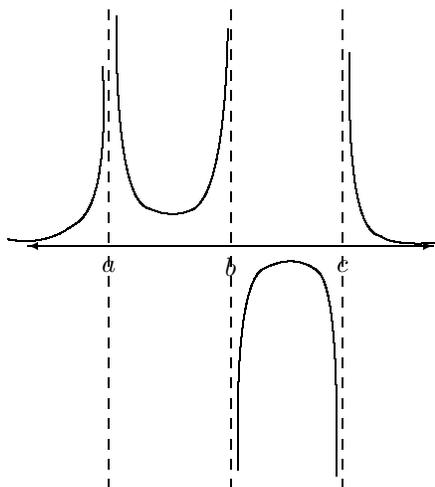
$$\lim_{x \rightarrow a^+} F(x) = \infty.$$

Similarly, if  $F(x)$  decreases without bound as  $x$  gets closer and closer to  $a$ , where  $x > a$ , we say that **the limit of  $F(x)$  as  $x$  approaches  $a$  from the right is minus infinity** and we write

$$\lim_{x \rightarrow a^+} F(x) = -\infty.$$

Left limits are defined similarly.

The unbounded behavior of a function  $F$  as  $x$  approaches  $a$  (from the right or from the left) is reflected in the graph via vertical asymptotes. The following picture illustrates the different possibilities, where the dotted lines  $x = a$ ,  $x = b$ , and  $x = c$  are vertical asymptotes:



We write

$$\lim_{x \rightarrow a^-} f(x) = \infty;$$

$$\lim_{x \rightarrow a^+} f(x) = \infty;$$

$$\lim_{x \rightarrow b^-} f(x) = \infty;$$

$$\lim_{x \rightarrow b^+} f(x) = -\infty;$$

$$\lim_{x \rightarrow c^-} f(x) = -\infty;$$

$$\lim_{x \rightarrow c^+} f(x) = \infty.$$

6. Answer Questions #8, #9, and #10. [Hint: For the theorem in Question #9, study the preceding examples carefully. Note that in the example in Question #9,  $x = 2$  is not a vertical asymptote. If you change the numerator to  $x - 1$ , is  $x = 2$  an asymptote?]

Your calculation should suggest that

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1}} = -1$$

**Definition 2.3** If  $\lim_{x \rightarrow \infty} f(x) = L$  or  $\lim_{x \rightarrow -\infty} f(x) = L$ , then the line  $y = L$  is a **horizontal asymptote** of  $f(x)$ .

Thus  $y = 1$  and  $y = -1$  are horizontal asymptotes for the function  $f(x) = \frac{x}{\sqrt{x^2 + 1}}$ .

7. To explore limits at infinity graphically, select Trace Mode and move the cross further and further to the right, and observe the behavior of the  $y$  coordinate of the cross. You may want to zoom out horizontally  to move the cross at a faster rate. The limit at minus infinity is likewise explored by moving the cross further and further to the left. Answer Questions #11 and #12.