Calculus Review

A. Functions:

1. Identify the domain and range for the following:

a) $f(x)=\sqrt{3-x^2} \Rightarrow D:\left[-\sqrt{3},\sqrt{3}\right] \Rightarrow R:\left[0,\sqrt{3}\right]$

b) $f(x)=\dfrac{3}{x^2-7x-8} \Rightarrow x^2-7x-8=0 \Rightarrow (x-8)(x+1)=0 \Rightarrow x=-1$ or 8

$D:(-\infty,-1)\cup(-1,8)\cup(8,\infty), R:(-\infty,0)\cup(0,\infty)$

2. Determine the equation given:

$D=(-\infty,-5)\cup(5,\infty), \quad R=[0,\infty) \Rightarrow$ upper half of a hyperbola

a) Form: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \Rightarrow \dfrac{x^2}{5^2}-\dfrac{y^2}{1^2}=1 \Rightarrow y=\dfrac{\sqrt{x^2-25}}{5}$

$D=(-\infty,-3)\cup(-3,2)\cup(2,\infty), \quad R=(-\infty,0)\cup(0,\infty)$

b) Rationl function with a constant in the numerator

$f(x)=\dfrac{1}{(x+3)(x-2)}$

3. Determine the slope of the tangent line to the curve $y=x^2-3x$. The slope formula must be developed using the slope formula and not differentiation.

$y=x^2-3x \Rightarrow$ Points $\Rightarrow (x, f(x)) \Rightarrow (x,(x^2-3x))$ and $\left((x+h),\left[(x+h)^2-3(x+h)\right]\right)$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{\left[(x+h)^2-3(x+h)\right]-(x^2-3x)}{(x+h)-x}=\frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}=$$

$$\frac{h^2+2xh-3h}{h}=\frac{h(h+2x-3)}{h}=h+2x-3 \Rightarrow \lim_{h\to 0}(h+2x-3)=2x-3$$

4. Given: $f(x)=x^2-5$ and $g(x)=x+1$, determine:

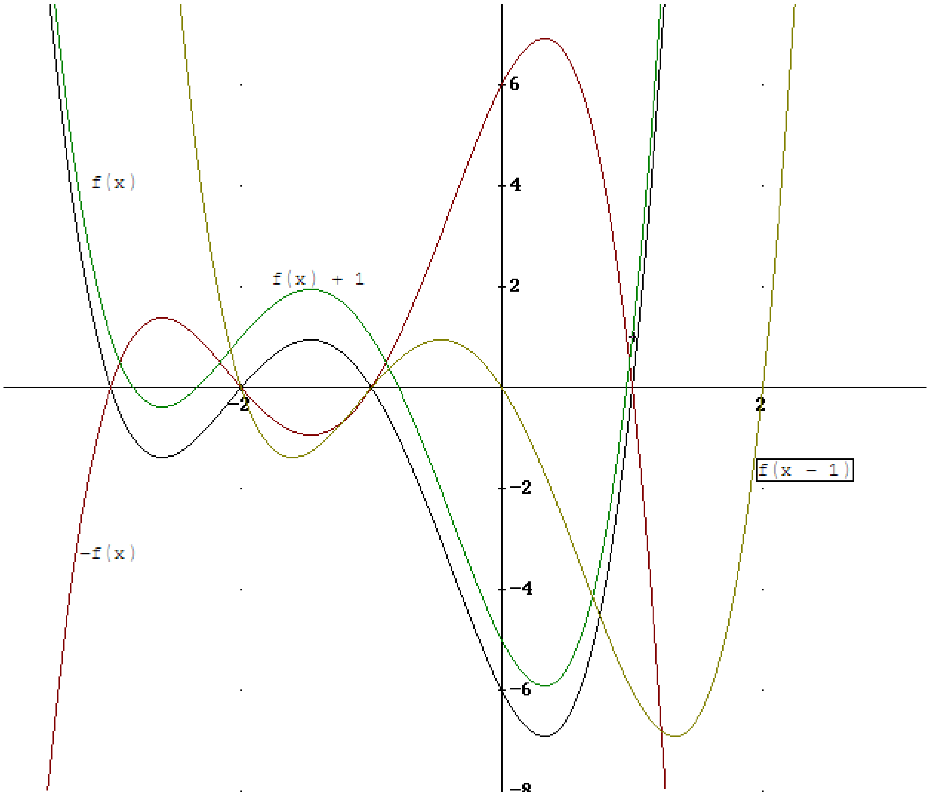
a) $f(-4) \Rightarrow f(x)=x^2-5 \Rightarrow f(-4)=(-4)^2-5=16-5=11$

b) $g(7) \Rightarrow g(x)=x+1 \Rightarrow g(7)=(7)+1=8$

c) $2f(x)-3g(x) \Rightarrow 2\left(x^2-5\right)-3(x+1)=2x^2-10-3x-3=2x^2-3x-13$

d) $(f\circ g)x \Rightarrow (x+1)^2-5=x^2+2x+1-5=x^2+2x-4$

5. Transformations:



Determine a) $-f(x)$, b) $f(x)+1$, c) $f(x-1)$

B. Limits

a) General

1. $\lim_{x\to 4}\sqrt{x+\sqrt{x}}=\sqrt{4+\sqrt{4}}=\sqrt{4+2}=\sqrt{6}$ 2. $\lim_{x\to -1}\frac{x+1}{x^2-x}=\frac{(-1)+1}{(-1)^2-(-1)}=\frac{0}{2}=0$

3.

$$\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x}=\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x}\cdot\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}=\lim_{x\to 0}\frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}=\lim_{x\to 0}\frac{x^2}{x(1+\sqrt{1-x^2})}=$$

$$\lim_{x\to 0}\frac{x}{(1+\sqrt{1-x^2})}=\frac{(0)}{(1+\sqrt{1-(0)^2})}=\frac{0}{2}=0$$

4. $\lim_{x\to -1}\frac{x^2-x-2}{x^2+3x+2}=\lim_{x\to -1}\frac{(x+1)(x-2)}{(x+1)(x+2)}=\lim_{x\to -1}\frac{(x-2)}{(x+2)}=\frac{((-1)-2)}{((-1)+2)}=\frac{-3}{1}=-3$

5.

$\lim_{x\to 0^-}\sqrt{-x}=0 \Rightarrow$ if a infinity small negative value is substituted into x,we are taking the square root of an infinity small positive number which tends to zero

6. $\lim_{x\to 2}\frac{x^2+2x-8}{x^4-16}=\lim_{x\to 2}\frac{(x+4)(x-2)}{(x^2+4)(x-2)(x+2)}=\lim_{x\to 2}\frac{(x+4)}{(x^2+4)(x+2)}=$

$\frac{((2)+4)}{((2)^2+4)((2)+2)}=\frac{6}{(8)(4)}=\frac{6}{32}=\frac{3}{16}$

7.

$\lim_{x\to\infty}\frac{2x^2-x-1}{4x^2+7}=\frac{2}{4}=\frac{1}{2}\Rightarrow$ since the degree of the numerator and denominator are the same the limit equals the ratio of the coefficents of the temrs containing the highest degree

or

$$\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}-\frac{x}{x^2}-\frac{1}{x^2}}{\frac{4x^2}{x^2}+\frac{7}{x^2}}=\lim_{x\to\infty}\frac{2-\frac{1}{x}-\frac{1}{x^2}}{4+\frac{7}{x^2}}=\frac{2-0-0}{4+0}=\frac{2}{4}=\frac{1}{2}$$

8.

$\lim_{x\to\infty}\frac{5x^3-7x^2}{4x^2+9x-5}=\infty\Rightarrow$ since the larger degree is found in the numerator the value of the limit tend to infinity

or

$$\lim_{x\to\infty}\frac{\frac{5x^3}{x^2}-\frac{7x^2}{x^2}}{\frac{4x^2}{x^2}+\frac{9x}{x^2}-\frac{5}{x^2}}=\lim_{x\to\infty}\frac{5x-7}{4+\frac{9}{x}-\frac{5}{x^2}}=\frac{\infty-7}{4+0-0}=\infty$$

9.

$\lim_{x\to -3^-}\frac{4x+1}{x+3}=\infty$

by substitution of $x=-3.001\Rightarrow \lim_{x\to -3^-}\frac{4x+1}{x+3}=\frac{4(-3.001)+1}{-3.001+3}=\frac{-11.001}{-0.001}=11001$

we can conclude that if we take a value infinity close to -3 on the left side and substitute into the expression, the value will tend towards plus infinity

10. $\lim_{x\to 0}\frac{\sin 5x}{3x}=\lim_{x\to 0}\frac{\sin 5x\cdot\frac{5}{3}}{3x\cdot\frac{5}{3}}=\frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{3}\cdot 1=\frac{5}{3}$

11.

$$\lim_{x\to0}\frac{\sec x}{1-\sin x}=\lim_{x\to0}\frac{\frac{1}{\cos x}}{1-\sin x}=\lim_{x\to0}\frac{1}{\cos x(1-\sin x)}\cdot\frac{(1+\sin x)}{(1+\sin x)}=\lim_{x\to0}\frac{(1+\sin x)}{\cos x(1-\sin^2 x)}=$$

$$\lim_{x\to0}\frac{(1+\sin x)}{\cos x\cos^2 x}=\lim_{x\to0}\frac{(1+\sin x)}{\left(\cos x\right)^3}=\frac{1+\sin 0}{\left(\cos 0\right)^3}=\frac{1+0}{1^3}=1$$

12.

$$\lim_{x\to0}\frac{\sin^2 3x}{x^2}=\lim_{x\to0}\frac{\sin 3x\cdot\sin 3x}{x\cdot x}=\lim_{x\to0}\frac{\sin 3x}{x}\cdot\frac{\sin 3x}{x}=\lim_{x\to0}\frac{\sin 3x}{x}\cdot\frac{3}{3}\cdot\frac{\sin 3x}{x}\cdot\frac{3}{3}=$$

$$9\lim_{x\to0}\frac{\sin 3x}{3x}\cdot\frac{\sin 3x}{3x}=9\cdot1\cdot1=9$$

b) Determine whether each of the following is continuous or discontinuous. If Discontinuous, determine whether a removable discontinuity exists and what it is.

a) $f(x)=-3x^3+6x-7$ -continuous

b) $f(x)=\dfrac{4}{9-x^2}$ discontinuous vertical asymptotes at 3 and -3

d) $f(x)=\dfrac{x^2-25}{x-5}$ removable discontinuity at x =5

C. Differentiation

a) General

1. $f(x)=x^3+5x+4\Rightarrow f'(x)=3x^2+5$

2. $$f(x)=\frac{4-x}{3+x}\Rightarrow f(x)=(4-x)(3+x)^{-1}\Rightarrow f'(x)=-1(3+x)^{-1}+(-1)(3+x)^{-2}(4-x)=$$

$$(3+x)^{-2}\left[-1(3+x)-1(4-x)\right]=(3+x)^{-2}(-7)$$

3. $f(x)=\sqrt{3-5x}\Rightarrow f(x)=(3-5x)^{\frac{1}{2}}\Rightarrow f'(x)=\dfrac{1}{2}(3-5x)^{-\frac{1}{2}}(-5)=-\dfrac{5}{2}(3-5x)^{-\frac{1}{2}}$

4. $f(x)=x\sin x\Rightarrow f'(x)=\sin x+x\cos x$

5. $$f(x)=\frac{x}{\sqrt{9-4x}}\Rightarrow f(x)=x\left(9-4x\right)^{-\frac{1}{2}}\Rightarrow f'(x)=\left(9-4x\right)^{-\frac{1}{2}}+-\frac{1}{2}\left(9-4x\right)^{-\frac{3}{2}}(-4)\cdot x=$$

$$\left(9-4x\right)^{-\frac{3}{2}}\left[\left(9-4x\right)+2x\right]=\left(9-4x\right)^{-\frac{3}{2}}(9-2x)$$

6. $f(x)=\sin(\cos x)\Rightarrow f'(x)=\cos(\cos x)\cdot-\sin x$

7. $f(x)=\sin x\cos x \Rightarrow f'(x)=\cos x\cos x+(-\sin x)\sin x=\cos^2 x-\sin^2 x=\cos(2x)$

8. $f(x)=(x^2-3)^4(3x-7)^5 \Rightarrow f'(x)=4(x^2-3)^3(2x)\cdot(3x-7)^5+5(3x-7)^4(3)\cdot(x^2-3)^4=$
$(3x-7)^4(x^2-3)^3\left[8x(3x-7)+15(x^2-3)\right]=(3x-7)^4(x^2-3)^3\left(39x^2-56x-45\right)$

9. $f(x)=e^{7x-5} \Rightarrow f'(x)=e^{7x-5}\cdot 7$

10. $f(x)=\ln(x^2-5)\cdot 7^{8x-1} \Rightarrow f'(x)=\dfrac{1}{(x^2-5)}\cdot 2x\cdot 7^{8x-1}+7^{8x-1}\cdot\ln 7\cdot 8\cdot\ln(x^2-5)=$
$7^{8x-1}\cdot 2\left(\dfrac{x}{(x^2-5)}+4\ln 7\ln(x^2-5)\right)$

11.

$$f(x)=\frac{\log_3(x^2-5x+1)}{e^{3x+1}} \Rightarrow f'(x)=\frac{\dfrac{1}{(x^2-5x+1)\ln 3}\cdot(2x-5)\cdot e^{3x+1}-e^{3x+1}\cdot 3\log_3(x^2-5x+1)}{\left(e^{3x+1}\right)^2}=$$

$$\frac{e^{3x+1}\left(\dfrac{1}{(x^2-5x+1)\ln 3}\cdot(2x-5)-3\log_3(x^2-5x+1)\right)}{\left(e^{3x+1}\right)^2}=\frac{\left(\dfrac{1}{(x^2-5x+1)\ln 3}\cdot(2x-5)-3\log_3(x^2-5x+1)\right)}{e^{3x+1}}$$

12.

$$f(x)=\frac{1}{\sin(x-\sin x)} \Rightarrow f(x)=\left(\sin(x-\sin x)\right)^{-1} \Rightarrow f'(x)=-1\left(\sin(x-\sin x)\right)^{-2}\cdot\cos(x-\sin x)\cdot(1-\cos x)$$

13. $f(x)=5x^3-2x^2+5x-3 \Rightarrow f'(x)=15x^2-4x+5$

14. $f(x)=(\ln(3x^2-5))^3 \Rightarrow f'(x)=3(\ln(3x^2-5))^2\cdot\dfrac{1}{(3x^2-5)}\cdot 6x$

15. $f(x)=7^{5x^3-3x+6} \Rightarrow f'(x)=7^{5x^3-3x+6}\cdot\left(15x^2-3\right)\cdot\ln 7$

16. $f(x)=\log_9(6x^5-7x) \Rightarrow f'(x)=\dfrac{1}{(6x^5-7x)\ln 9}\cdot\left(30x^4-7\right)$

17. $f(x)=e^{5x-7} \Rightarrow f'(x)=e^{5x-7}\cdot 5$

18. $f(x)=(\cos(6x^3-2x+1))^3 \Rightarrow f'(x)=3(\cos(6x^3-2x+1))^2\cdot -\sin(6x^3-2x+1)\cdot\left(18x^2-2\right)$

b) Higher Order

1. - 3^{rd} order derivative

$$\begin{aligned} f(x)&=4x^3-3x^2-18x+5\\ f'(x)&=12x^2-6x-18\\ f''(x)&=24x-6\\ f'''(x)&=24 \end{aligned}$$

2. - 2^{nd} order derivative

$$\begin{aligned} f(x)&=\sin^2 x\cos x\\ f'(x)&=2\sin x(\cos x)\cdot\cos x+(-\sin x)\cdot\sin^2 x\\ f'(x)&=2\sin x\cos^2 x-\sin^3 x\\ f''(x)&=2\cos x\cdot\cos^2 x+2\cos x(-\sin x)\cdot 2\sin x-3\sin^2 x(\cos x)\\ f''(x)&=2\cos^3 x-4\sin^2 x\cos x-3\sin^2 x(\cos x)\\ f''(x)&=2\cos^3 x-7\sin^2 x\cos x \end{aligned}$$

3. If $f(x)=(2-x^2)^6$, find f(0), f'(0), f''(0)

$$f(x)=(2-x^2)^6 \Rightarrow f(0)=(2-(0)^2)^6=2^6=64$$

$$f'(x)=6(2-x^2)^5(-2x)\Rightarrow f'(x)=-12x(2-x^2)^5\Rightarrow$$

$$f(0)=6(2-(0)^2)^5(-2(0))=0$$

,

$$f''(x)=-12(2-x^2)^5+5(2-x^2)^4(-2x)\cdot -12x\Rightarrow -12(2-x^2)^4\left[(2-x^2)-10x^2\right]=$$

$$-12(2-x^2)^4\left[2-11x^2\right]\Rightarrow f''(0)=-12(2-(0)^2)^4\left[2-11(0)^2\right]=-12(16)(2)=384$$

c) Implicit Differentiation

1.

$$x^2y+xy^3=2\Rightarrow 2x\cdot y+x^2\cdot\frac{dy}{dx}+y^3+3y^2\frac{dy}{dx}\cdot x=0\Rightarrow$$

$$x^2\frac{dy}{dx}+3xy^2\frac{dy}{dx}=-2xy-y^3\Rightarrow\frac{dy}{dx}=\frac{-2xy-y^3}{x^2+3xy^2}$$

2.

$$\ln(x^2+1)+8xy-e^{2y}=0\Rightarrow\frac{1}{\left(x^2+1\right)}\cdot 2x+8\cdot y+\frac{dy}{dx}\cdot 8x-e^{2y}\cdot 2\frac{dy}{dx}=0\Rightarrow$$

$$8x\frac{dy}{dx}-2e^{2y}\frac{dy}{dx}=\frac{-2x}{\left(x^2+1\right)}+8y\Rightarrow\frac{dy}{dx}=\frac{\frac{-2x}{\left(x^2+1\right)}+8y}{8x-2e^{2y}}=\frac{-2x+8y\left(x^2+1\right)}{\left(x^2+1\right)\left(8x-2e^{2y}\right)}$$

3. $\sqrt[3]{x}-\sqrt{y}=2 \Rightarrow x^{\frac{1}{3}}-y^{\frac{1}{2}}=2 \Rightarrow \frac{1}{3}x^{-\frac{2}{3}}-\frac{1}{2}y^{-\frac{1}{2}}\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{\frac{1}{3}x^{-\frac{2}{3}}}{-\frac{1}{2}y^{-\frac{1}{2}}}=-\frac{2}{3}\frac{x^{-\frac{2}{3}}}{y^{-\frac{1}{2}}}$

D. Integration

a) Definite Integral

1. $\int_{-3}^{0}(2x^3-3x-4)dx \Rightarrow 2\frac{x^4}{4}-3\frac{x^2}{2}-4x\Big|_{-3}^{0}=$
$\left[2\frac{(0)^4}{4}-3\frac{(0)^2}{2}-4(0)\right]-\left[2\frac{(-3)^4}{4}-3\frac{(-3)^2}{2}-4(-3)\right]=-39$

2. $\int_{0}^{1}(5\cos x+4x)dx=5\sin x+4\frac{x^2}{2}\Big|_{0}^{1}=\left[5\sin(1)+2(1)^2\right]-\left[5\sin(0)+2(0)^2\right]=6.20$

3. $\int_{1}^{\sqrt{3}}\frac{6}{1+x^2}dx=6\tan^{-1}x\Big|_{1}^{\sqrt{3}}=\left[6\tan^{-1}\sqrt{3}\right]-\left[6\tan^{-1}1\right]=\frac{\pi}{2}$

b) General

1. $\int 5dx=5x$

2. $\int(3x-7)dx=3\frac{x^2}{2}-7x$

3. $\int\sqrt{x}dx=\int x^{\frac{1}{2}}dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2x^{\frac{3}{2}}}{3}$

4. $\int\sqrt{x}-\frac{2}{\sqrt{x}}dx \Rightarrow \int x^{\frac{1}{2}}-2x^{\frac{-1}{2}}dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-2\frac{x^{\frac{1}{2}}}{\frac{1}{2}}=\frac{2x^{\frac{3}{2}}}{3}-4x^{\frac{1}{2}}$

5. $\int\left(x+\frac{1}{x}\right)^2dx=\int x^2+2+\frac{1}{x^2}dx=\int x^2+2+x^{-2}dx=\frac{x^3}{3}+2x-x^{-1}$

6. $\int\frac{1}{x^2+36}dx=\int\frac{1}{x^2+6^2}dx=\frac{1}{6}\tan^{-1}\frac{x}{6}$

7. $$\int \frac{x+3}{(x^2+6x)^2}dx = \frac{1}{2}\int \frac{1}{u^2}du = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\cdot\frac{u^{-1}}{-1} = -\frac{1}{2u} = -\frac{1}{2\left(x^2+6x\right)}$$

$$u = x^2+6x \Rightarrow du = \left(2x+6\right)dx \Rightarrow \frac{1}{2}du = (x+3)dx$$

8. $$\int \ln(5x^2-3)xdx = \frac{1}{10}\int \ln u du = \frac{1}{10}(u\ln u - u) = \frac{1}{10}\left[\left(5x^2-3\right)\ln\left(5x^2-3\right)-\left(5x^2-3\right)\right]$$

$$u = 5x^2-3 \Rightarrow du = 10xdx \Rightarrow \frac{1}{10du} = xdx$$

9. $$\int 7^{x^2-7}2xdx = \int 7^u du = \frac{7^u}{\ln 7} = \frac{7^{x^2-7}}{\ln 7}$$

$$u = x^2-7 \Rightarrow du = 2xdx$$

10. $$\int e^x \sin(e^x)dx = \int \sin u du = -\cos u = -\cos\left(e^x\right)$$

$$u = e^x \Rightarrow du = e^x dx$$

11. $$\int \frac{\sin x}{1+\cos^2 x}dx = -\int \frac{1}{1+u^2}du = -\tan^{-1}u = -\tan^{-1}\left(\cos x\right)$$

$$u = \cos x \Rightarrow du = -\sin xdx$$

12. $$\int \frac{1}{x\sqrt{\ln x}}dx = \int \frac{1}{x\left(\ln x\right)^{\frac{1}{2}}}dx = \int \frac{1}{x}\cdot\left(\ln x\right)^{-\frac{1}{2}}dx = \int u^{-\frac{1}{2}}du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = 2\left(\ln x\right)^{\frac{1}{2}}$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

13.

$$\int x^2 \sin 2x dx = -\frac{1}{2}\cos(2x)\cdot x^2 - \int 2x \cdot -\frac{1}{2}\cos(2x)dx = -\frac{1}{2}\cos(2x)\cdot x^2 + \int x \cdot \cos(2x)dx =$$

$$u = x^2 \Rightarrow du = 2xdx$$

$$dv = \int \sin 2xdx \Rightarrow v = -\frac{1}{2}\cos(2x)$$

$$-\frac{1}{2}x^2\cos(2x) + \left[\frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)dx\right] = -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)dx =$$

$$u = x \Rightarrow du = dx$$

$$dv = \int \cos(2x)dx \Rightarrow v = \frac{1}{2}\sin(2x)$$

$$-\frac{1}{2}\cos(2x)\cdot x^2 + \frac{1}{2}x\sin(2x) - \frac{1}{2}\cdot\frac{1}{2}\int \sin u du = -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\cdot\frac{1}{2}(-\cos u) =$$

$$u = 2x \Rightarrow du = 2dx \Rightarrow \frac{1}{2}du = dx$$

$$-\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) =$$

14. $\int \cos x \ln(\sin x)dx = \int \ln u = u\ln u - u = \sin x \ln\left(\sin x\right) - \sin x$

$u = \sin x \Rightarrow du = \cos x dx$

15.

$$\int \frac{3x^2-6x+2}{2x^3-3x^2+x}dx =$$

$$x(2x^2-3x+1) \Rightarrow x(2x-1)(x-1) \Rightarrow \frac{A}{x}+\frac{B}{(2x-1)}+\frac{C}{(x-1)} \Rightarrow$$

$$\frac{A(2x-1)(x-1)+Bx(x-1)+Cx(2x-1)}{x(2x-1)(x-1)} \Rightarrow$$

$$A(2x^2-3x+1)+B(x^2-x)+C(2x^2-x) = 2Ax^2-3Ax+A+Bx^2-Bx+2Cx^2-Cx \Rightarrow$$

$$(2A+B+2C)x^2+(-3A-B-C)+A$$

$2A+B+2C=3$ by substitution $2(2)+B+2C=3 \Rightarrow B+2C=-1$

$-3A-B-C=-6$ by substitution $-3(2)-B-C=-6 \;\Rightarrow -B-C=0 \Rightarrow B=-C$

$A=2$

Solving the system of equations: $B+2C=-1$ and $-B-C=0 \Rightarrow B=-C$

$-C+2C=-1 \Rightarrow C=-1$ and therefore $B=1$

$$\int \frac{2}{x}+\frac{1}{(2x-1)}+\frac{-1}{(x-1)}dx \Rightarrow \int \frac{2}{x}dx+\int \frac{1}{(2x-1)}dx-\int \frac{1}{(x-1)}dx = 2\ln x+\frac{1}{2}\ln(2x-1)-\ln(x-1)$$

$$u=2x-1 \Rightarrow du=2dx \Rightarrow \frac{1}{2}du=dx$$

$$\int \frac{1}{(2x-1)}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln u = \frac{1}{2}\ln(2x-1)$$

E. Curve Sketching:

$f(x) = x^4 - 6x^2$

x-intercept $\Rightarrow x^4 - 6x^2 = 0 \Rightarrow x^2(x^2 - 6) = 0 \Rightarrow x = 0, x = \pm\sqrt{6}$

y-intercept $\Rightarrow y = 0$

$f'(x) = 4x^3 - 12x \Rightarrow 4x(x^2 - 3) = 0 \Rightarrow x = 0, x = \pm\sqrt{3}$

max and min value occur at $(0,0), (\sqrt{3}, -9), (-\sqrt{3}, -9)$, using intervals

$(-\infty, -\sqrt{3}), \Rightarrow$ negative value for k = -2 $\Rightarrow$ graph decreasing

$(-\sqrt{3}, 0) \Rightarrow$ positive value for k= -1 $\Rightarrow$ graph increasing

$(0, \sqrt{3},) \Rightarrow$ negative value for k= 1 $\Rightarrow$ graph decreasing

$(\sqrt{3}, \infty) \Rightarrow$ positive value for k= -1 $\Rightarrow$ graph increasing

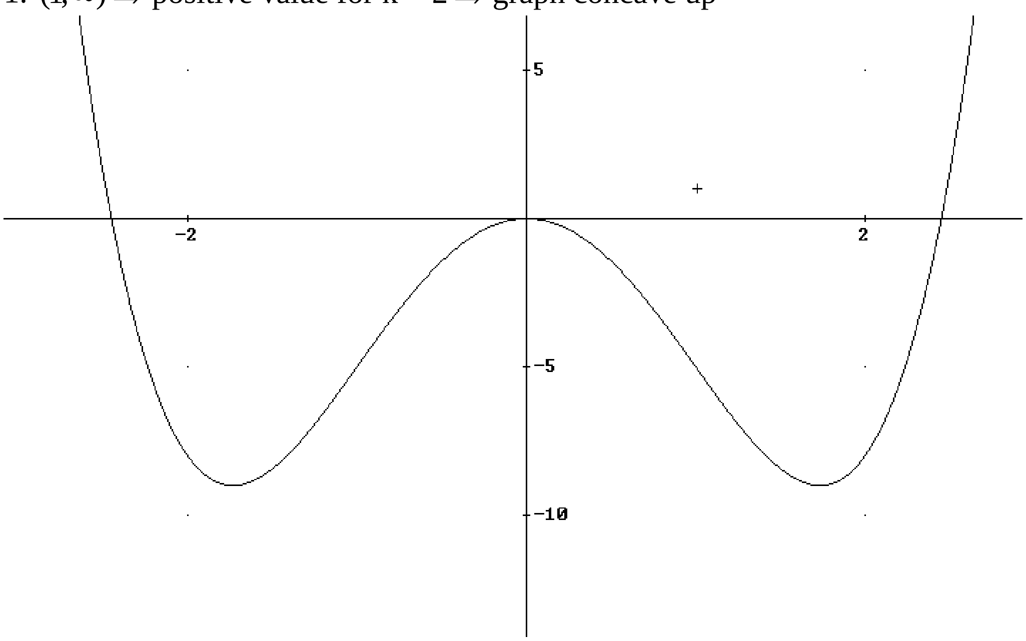
$f''(x) = 12x^2 - 12 \Rightarrow 12(x^2 - 1) = 0 \Rightarrow x = \pm 1$

Points of inflection occur at $(-1, -5)$ and $(1, -5)$, using intervals

$(-\infty, -1) \Rightarrow$ positive value for k = -2 $\Rightarrow$ graph concave up

$(-1, 1) \Rightarrow$ negative value for k = 0 $\Rightarrow$ graph concave down

1. $(1, \infty) \Rightarrow$ positive value for k = 2 $\Rightarrow$ graph concave up



2.

$f(x)=\frac{1}{x^2(x+3)} \Rightarrow f(x)=\frac{1}{(x^3+3x)} \Rightarrow f(x)=(x^3+3x^2)^{-1}$

vertical asymptotes at x = 0 and x = -3

x-intercept $\Rightarrow$ undefined for $y=0$, therefore no x-intercepts

y-intercept $\Rightarrow$ *none* since we have a vertical asymptote at x = 0 or the y-axis

$f\,'(x)=-1(x^3+3x^2)^{-2}(3x^2+6x) \Rightarrow -1x^{-4}(x+3)^{-2}\cdot 3x(x+2) \Rightarrow -3x^{-3}(x+3)^{-2}(x+2)$

the only solution is for the factor $(x+2) \Rightarrow x=-2,$

therfore a min value at point (-2, 1/4) based on cacluations below

because of the asymptotes the intervals we will consider include:

$(-\infty,-3) \Rightarrow$ negative result at k = -4 $\Rightarrow$ graph decreasing

$(-3,-2) \Rightarrow$ negative result at k = -2.5 $\Rightarrow$ graph decreasing

$(-2,0) \Rightarrow$ positive result at k = -1.5 $\Rightarrow$ graph increasing

$(0,\infty) \Rightarrow$ negative result at k = 1 $\Rightarrow$ graph decreasing

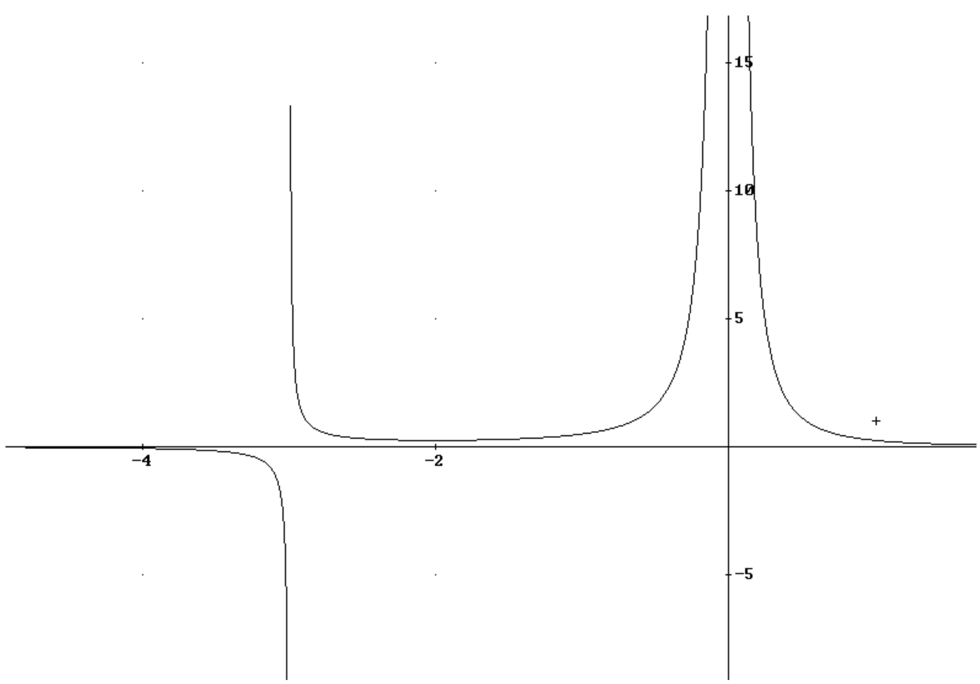
$f\,''(x)=-1(-2)(x^3+3x^2)^{-3}(3x^2+6x)\cdot(3x^2+6x)+(6x+6)(-1(x^3+3x^2)^{-2})=$

$2x^{-6}(x+3)^{-3}\cdot 3x(x+2)\cdot 3x(x+2)-6(x+1)\cdot x^{-4}(x+3)^{-2}=$

$18x^{-4}(x+3)^{-3}(x+2)(x+2)-6x^{-4}(x+1)(x+3)^{-2}=6x^{-4}(x+3)^{-3}\left[3(x+2)(x+2)-(x+1)(x+3)\right]=$

$6x^{-4}(x+3)^{-3}(2x^2+8x+9)$

no solutions therefore no calculated inflection points



F. Problem Solving (Related Rates)

1. For $s(t) = t^3 - 3t^2 + 5$ determine:
   a) velocity at t = 2,
   b) acceleration at t = 2,
   c) maximum height reached,
   d) d) time it takes to reach the ground,
   e) total distance traveled.

$$s(t) = t^3 - 3t^2 + 5 \Rightarrow velocity \Rightarrow s'(t) = 3t^2 - 6t \Rightarrow acceleration \Rightarrow s''(t) = 6t - 6$$

$$a)\ s'(2) = 3(2)^2 - 6(2) = 0$$

$$b)\ s''(2) = 6(2) - 6 = 6$$

$$c)\ \text{max height} = s(t) = (2)^3 - 3(2)^2 + 5 = 1$$

$$d)\ \text{time} = 2, \text{total time} = 4$$

$$e)\ \text{total distance} = 2$$

2. Find the equation of a line tangent to the curve $f(x) = 2x^3 - 4x + 1$ at the point having an x-coordinate of -2.

$$f(x) = 2x^3 - 4x + 1 \Rightarrow f(2) = 2(2)^3 - 4(2) + 1 = 9 \Rightarrow (2,9)$$

$$m = f'(x) = 6x^2 - 4 \Rightarrow m = 6(2)^2 - 4 = 20$$

$$(y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y - 9) = 20(x - 2) \Rightarrow y - 9 = 20x - 40 \Rightarrow y = 20x - 31$$

2. What is the slope of the line tangent to the curve $x^3 + 2x^2y + y = 5$ at the point (-1, 2)

$$m = \frac{dy}{dx}$$

$$x^3 + 2x^2y + y = 5 \Rightarrow 3x^2 + 4xy + 2x^2\frac{dy}{dx} + \frac{dy}{dx} = 0 \Rightarrow 2x^2\frac{dy}{dx} + \frac{dy}{dx} = -3x^2 - 4xy \Rightarrow$$

$$\frac{dy}{dx} = \frac{-3x^2 - 4xy}{2x^2 + 1} = \frac{-3(-1)^2 - 4(-1)(2)}{2(-1)^2 + 1} = \frac{-3 + 8}{3} = \frac{5}{3}$$

4. A spherical balloon is being inflated at a rate of 10 cubic meters per minute. Find the rate at which the radius is increasing a) when the radius is 5m, b) when the volume is 36 meters cubed.

$$a)\ V=\frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt}=\frac{4}{3}\pi 3r^2\frac{dr}{dt}\Rightarrow 10=4\pi(5)^2\frac{dr}{dt}\Rightarrow 10=100\pi\frac{dr}{dt}\Rightarrow \frac{10}{100\pi}=\frac{dr}{dt}$$

$$b)\ V=\frac{4}{3}\pi r^3 \Rightarrow 36=\frac{4}{3}\pi r^3\Rightarrow \frac{108}{4\pi}=r^3\Rightarrow \frac{27}{\pi}=r^3\Rightarrow \left(\frac{27}{\pi}\right)^{\frac{1}{3}}=r$$

$$\frac{dV}{dt}=\frac{4}{3}\pi 3r^2\frac{dr}{dt}\Rightarrow 10=4\pi(\left(\frac{27}{\pi}\right)^{\frac{1}{3}})^2\frac{dr}{dt}\Rightarrow 10=4\pi\left(\frac{27}{\pi}\right)^{\frac{2}{3}}\frac{dr}{dt}\Rightarrow \frac{10}{4\pi\left(\frac{27}{\pi}\right)^{\frac{2}{3}}}=\frac{dr}{dt}$$

5. A ladder 8m long is leaning against a wall. The bottom of the ladder is sliding away from the wall at 1.5 m/s. At what rate is the top of the ladder sliding down the wall at the instant when the bottom of the ladder is 5 meters from the wall?

$$c^2=a^2+b^2\Rightarrow c\frac{dc}{dt}=a\frac{da}{dt}+b\frac{db}{dt}\text{ and }8^2=a^2+5^2\Rightarrow a=\sqrt{39}$$

$$c=8,\frac{dc}{dt}=0,a=\sqrt{39},\frac{da}{dt},b=5,\frac{db}{dt}=1.5$$

$$8\cdot 0=\sqrt{39}\cdot\frac{da}{dt}+5\cdot 1.5\Rightarrow 0=\sqrt{39}\cdot\frac{da}{dt}+7.5\Rightarrow -\frac{7.5}{\sqrt{39}}=\frac{da}{dt}$$

6. Crushed gravel is being unloaded from a conveyor belt and as it is being poured the gravel forms a conical pile whose base radius is increasing as its height is increasing. If the base radius is increasing at 0.2 m/min and the height is increasing at 0.3 m/min, find the rate at which the volume is increasing?

$$V=\frac{1}{3}\pi r^2h\Rightarrow V=\frac{1}{3}\pi r^2\left(\frac{3r}{2}\right)\Rightarrow V=\frac{1}{2}\pi r^3\Rightarrow\frac{dV}{dt}=\frac{1}{2}\pi\cdot 3r^2\frac{dr}{dt}\Rightarrow\frac{dV}{dt}=\frac{3}{2}\pi r^2\frac{dr}{dt}$$

$$\text{ratio of }\frac{r}{h}=\frac{.2}{.3}=,\frac{2}{3}\Rightarrow\frac{3r}{2}=h$$

$$\text{if }r=1,\frac{dr}{dt}=.2,\frac{dV}{dt}=?\Rightarrow\frac{dV}{dt}=\frac{3}{2}\pi(1)^2.2\Rightarrow\frac{dV}{dt}=.3\pi$$

7. Water is being poured into a conical tank at a rate of 30 cubic meters per minute. If the height and radius at the top of the tank are 12m and 8 meters respectively, find the rate at which the water level is rising at the instant when the height is 4m.

$$V=\frac{1}{3}\pi r^2h\Rightarrow V=\frac{1}{3}\pi\left(\frac{2}{3}h\right)^2h\Rightarrow V=\frac{4}{27}\pi h^3\Rightarrow$$

$$ratio\Rightarrow\frac{h}{r}=\frac{12}{8}\Rightarrow\frac{8h}{12}=r\Rightarrow\frac{2}{3}h=r$$

$$\frac{dV}{dt}=\frac{4}{27}\pi\cdot 3h^2\frac{dh}{dt}\Rightarrow\frac{dV}{dt}=\frac{4}{9}\pi h^2\frac{dh}{dt}\Rightarrow 30=\frac{4}{9}\pi(4)^2\frac{dh}{dt}\Rightarrow\frac{270}{64\pi}=\frac{dh}{dt}$$

8. Two ships leave port at the same time. Ship a travels west at 20km, while ship B heads south at 35 km. At what rate are the ships separating after one hour?

$c^2 = a^2 + b^2 \Rightarrow c\frac{dc}{dt} = a\frac{da}{dt} + b\frac{db}{dt}$ and $c^2 = 20^2 + 35^2 \Rightarrow c = \sqrt{1625}$

$c = \sqrt{1625}, \frac{dc}{dt} = ?, a = 20, \frac{da}{dt} = 20, b = 35, \frac{db}{dt} = 35$

$\sqrt{1625}\frac{dc}{dt} = 20 \cdot 20 + 35 \cdot 35 \Rightarrow \sqrt{1625}\frac{dc}{dt} = 1625 \Rightarrow \frac{dc}{dt} = \frac{1625}{\sqrt{1625}} = \sqrt{1625}$

G. Optimization:

1. A piece of wire 8cm long is cut into two pieces. One piece is bent to form a circle and the other is bent to form a
   square.

   $x + y = 8 \Rightarrow y = 8 - x$

   $C = 2\pi r \Rightarrow x = 2\pi r \Rightarrow \frac{x}{2\pi} = r$

   $Perimeter = 8 - x \Rightarrow$ each side $= \frac{8-x}{4}$

   $MA = \pi r^2 + s^2 \Rightarrow MA = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{8-x}{4}\right)^2 \Rightarrow MA = \pi\frac{x^2}{4\pi^2} + \frac{64-16x+x^2}{16} \Rightarrow$

   $MA = \frac{x^2}{4\pi} + \frac{64-16x+x^2}{16} \Rightarrow MA' = \frac{1}{4\pi}\cdot 2x + \frac{1}{16}(-16+2x) \Rightarrow \frac{1}{4\pi}\cdot 2x + \frac{1}{16}(-16+2x) = 0$

   $\Rightarrow \frac{1}{2\pi}x - 1 + \frac{1}{8}x = 0 \Rightarrow 4x - 8\pi + \pi x = 0 \Rightarrow x(4+\pi) = 8\pi \Rightarrow x = \frac{8\pi}{(4+\pi)} = 3.51$

   0r

Perimeter of the square = $4x$, Circumference of circle = $2\pi y$

Perimeter + Circumference = Total length of wire

Solving for the radius of the circle:

$8 = 4x + 2\pi y \Rightarrow x = \frac{8-2\pi y}{4} = \frac{8}{4} - \frac{\pi y}{2}$

$A = x^2 + \pi y^2 \Rightarrow A = \left(\frac{8}{4} - \frac{\pi y}{2}\right)^2 + \pi y^2 \Rightarrow$

$A' = 2\left(\frac{8}{4} - \frac{\pi y}{2}\right)\cdot\left(\frac{-\pi}{2}\right) + 2\pi y \Rightarrow A' = \frac{-\pi 8}{4} + \frac{\pi^2 y}{2} + 2\pi y \Rightarrow$

$0 = \frac{-\pi 8}{4} + \frac{\pi^2 y}{2} + 2\pi y \Rightarrow \frac{\pi 8}{4} = \pi y\left(\frac{\pi}{2} + 2\right) \Rightarrow \frac{\pi 8}{4} = \pi y\left(\frac{\pi + 4}{2}\right) \Rightarrow$

$\frac{\pi 8}{4}\cdot\frac{2}{(\pi+4)} = \pi y \Rightarrow \frac{\pi 8}{2\pi}\cdot\frac{1}{(\pi+4)} = y \Rightarrow y = \frac{8}{(2\pi+8)}$

evaluate y then substitute into the circumference formula = 3.51

2. An open top box is to be made by cutting a square from the corners of a 12 inch by 12 inch sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible.

$V(x) = (12-2x)(12-2x)x = 144x - 48x^2 + 4x^3$

$Domain: 0 \le x \le 6$

$V'(x) = 144 - 96x + 12x^2 \Rightarrow 12(12 - 8x + x^2) = 12(2-x)(6-x)$

$Zeros: x = 2, x = 6$

$V(2) = 144x - 48x^2 + 4x^3 \Rightarrow 144(2) - 48(2)^2 + 4(2)^3 = 128$

3. Design a 1 liter oil can shaped like a right cylinder. What dimensions will use the least amount of material?

$V = \pi r^2 h = 1000 \Rightarrow \frac{1000}{\pi r^2} = h$

area of circle = πr^2, and area of sides = $2\pi rh$

$SA = 2\pi r^2 + 2\pi rh \Rightarrow SA = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \Rightarrow SA = 2\pi r^2 + 2000r^{-1} \Rightarrow$

$SA' = 4\pi r - 2000r^{-2} \Rightarrow 0 = 4\pi r - 2000r^{-2} \Rightarrow 4\pi r^3 = 2000 \Rightarrow$

$r = \sqrt[3]{\frac{2000}{4\pi}} = 5.42$

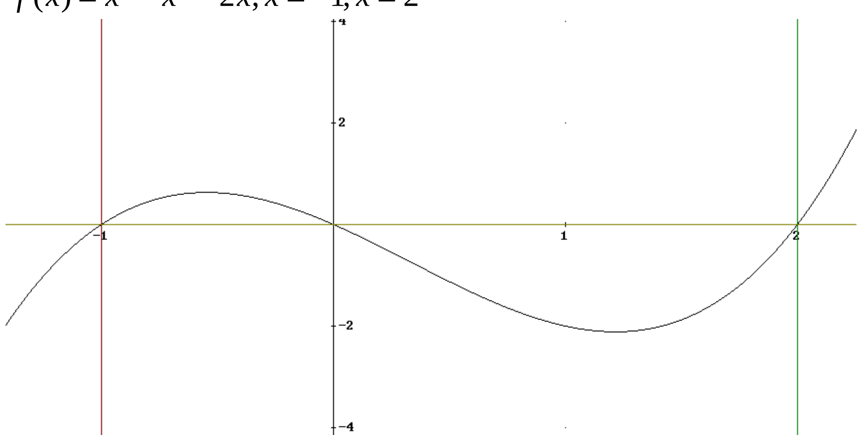
$Domain:$ open interval, therefore require a 2nd derivative

$SA'' = 4\pi + \frac{4000}{r^3} \Rightarrow$ postive throughout the domain $\Rightarrow$ graph is concave up therefore a minimum

$h = 2r \Rightarrow h = 10.84$

H. Area between Curves

1. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x, x = -1, x = 2$



Factor $\Rightarrow x(x-2)(x+1) \Rightarrow x=0, x=2, x=-1$

Integral over [-1,0]

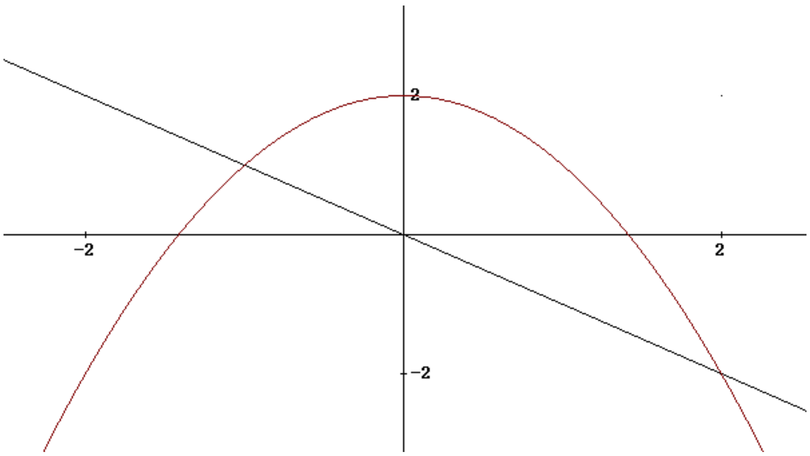
$$\int_{-1}^{0}(x^3-x^2-2x)-(0)dx = \frac{x^4}{4}-\frac{x^3}{3}-2\frac{x^2}{2}\Bigg|_{-1}^{0} = \frac{5}{12}$$

Integral over [0,2]

$$\int_{-1}^{0}(0)-(x^3-x^2-2x)dx = -\frac{x^4}{4}+\frac{x^3}{3}+2\frac{x^2}{2}\Bigg|_{0}^{2} = \frac{8}{3}$$

Total Area $=\frac{37}{12}$

3. Find the area of the region enclosed by the parabola $y=2-x^2$ and the line $y=-x$



Points of intersection:

$2-x^2=-x \Rightarrow x^2-x-2=0 \Rightarrow (x-2)(x+1)=0 \Rightarrow x=-1, x=2$

$$\int_{-1}^{2}\left(2-x^2\right)-(-x)dx = \int_{-1}^{2}-x^2+x+2dx = -\frac{x^3}{3}+\frac{x^2}{2}+2x\Bigg|_{-1}^{2} =$$

$$\left(-\frac{(2)^3}{3}+\frac{(2)^2}{2}+2(2)\right)-\left(-\frac{(-1)^3}{3}+\frac{(-1)^2}{2}+2(-1)\right)=\frac{9}{2}$$