Limits at Infinity and of Trig Functions

1. $$\lim_{x \to 0} \frac{3\sin x}{x} = 3 \cdot \lim_{x \to 0} \frac{\sin x}{x} = 3 \cdot 1 = 3$$

2. $$\lim_{x \to 0} \frac{\cos x - 1}{\tan x} = \lim_{x \to 0} \frac{\cos x - 1}{\frac{\sin x}{\cos x}} = \lim_{x \to 0} \frac{\cos x\left(\cos x - 1\right)}{\sin x} \cdot \frac{x}{x} =$$

$$\lim_{x \to 0} \cos x \cdot \frac{\cos x - 1}{x} \cdot \frac{x}{\sin x} = 1 \cdot 0 \cdot 1 = 0$$

3. $$\lim_{x \to 0} \frac{\sin(5x)}{\sin(2x)} = \lim_{x \to 0} \frac{\sin(5x)}{\sin(2x)} \cdot \frac{5x}{5x} \cdot \frac{2x}{2x} = \lim_{x \to 0} \frac{\sin(5x)}{5x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{5x}{2x} = 1 \cdot 1 \cdot \frac{5}{2} = \frac{5}{2}$$

4. $$\lim_{x \to \infty} \frac{-7x}{\sqrt{4x^2 + 3}} = \lim_{x \to \infty} \frac{\frac{-7x}{x}}{\sqrt{\frac{4x^2}{x^2} + \frac{3}{x^2}}} = \lim_{x \to \infty} \frac{-7}{\sqrt{4 + \frac{3}{x^2}}} = \frac{-7}{2}$$

$$\lim_{x \to \infty} \sqrt{x^2 + x + 2} - x = \lim_{x \to \infty} \frac{\sqrt{x^2 + x + 2} - x}{1} \cdot \frac{\sqrt{x^2 + x + 2} + x}{\sqrt{x^2 + x + 2} + x} =$$

5. $$\lim_{x \to \infty} \frac{x + 2}{\sqrt{x^2 + x + 2} + x} = \lim_{x \to \infty} \frac{\frac{x}{x} + \frac{2}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{x}{x^2} + \frac{2}{x^2}} + \frac{x}{x}} = \lim_{x \to \infty} \frac{1 + \frac{2}{x}}{\sqrt{1 + \frac{1}{x} + \frac{2}{x^2}} + 1} =$$

$$\frac{1}{\sqrt{1} + 1} = \frac{1}{2}$$

6. $$\lim_{x \to 0} \frac{\sin 4x}{3x} = \lim_{x \to 0} \frac{\sin 4x}{3x} \cdot \frac{4}{4} = \frac{4}{3} \cdot \lim_{x \to 0} \frac{\sin 4x}{4x} = \frac{4}{3} \cdot 1 = \frac{4}{3}$$

7.

$$\lim_{x \to 0} \frac{x+\cos x}{\sin x} = \lim_{x \to 0} \frac{x+\cos x}{\sin x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{x}{x}+\frac{\cos x}{x}}{\frac{\sin x}{x}} =$$

$$\lim_{x \to 0} \left(1+\frac{\cos x}{x}\right) \cdot \frac{x}{\sin x} = \left(1+undefined\right) \cdot 1 = undefined$$

8.

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x \to 0} \frac{1-\cos^2 x}{x^2\left(1+\cos x\right)} =$$

$$\lim_{x \to 0} \frac{\sin^2 x}{x^2\left(1+\cos x\right)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{1+\cos x} = 1 \cdot 1 \cdot \frac{1}{1+1} = \frac{1}{2}$$

9.

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot 1 = 1$$

$$\lim_{x \to 0} \frac{7x\cos x + 3\sin x}{3x^2 + \tan x} = \lim_{x \to 0} \frac{\left(7x\cos x + 3\sin x\right) \cdot \frac{\sin x}{\sin x}}{\left(3x^2 + \tan x\right) \cdot \frac{x}{x}} = \lim_{x \to 0} \frac{\sin x\left(\frac{7x\cos x}{\sin x} + \frac{3\sin x}{\sin x}\right)}{x\left(\frac{3x^2}{x} + \frac{\tan x}{x}\right)} =$$

$$\lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\left(7x\cot x + 3\right)}{\left(3x + \frac{\tan x}{x}\right)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\left(\frac{7x}{\tan x} + 3\right)}{\left(3x + \frac{\tan x}{x}\right)} = 1 \cdot \left(\frac{7(1)+3}{3(0)+1}\right) = \frac{10}{x}$$

10.

$$\lim_{x \to 0} \frac{5x + 7\sin x}{7x + 5\sin x} = \lim_{x \to 0} \frac{\left(5x + 7\sin x\right) \cdot \frac{1}{x}}{\left(7x + 5\sin x\right) \cdot \frac{1}{x}} = \lim_{x \to 0} \frac{\frac{5x}{x} + \frac{7\sin x}{x}}{\frac{7x}{x} + \frac{5\sin x}{x}} =$$

$$\lim_{x \to 0} \frac{5 + 7 \cdot \frac{\sin x}{x}}{7 + 5 \cdot \frac{\sin x}{x}} = \frac{5 + 7 \cdot 1}{7 + 5 \cdot 1} = \frac{12}{12} = 1$$

11. $$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}=$$
$$\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}=1\cdot 1\cdot\frac{1}{1+1}=\frac{1}{2}$$

Half angle formula : $\sin\frac{x}{2}=\sqrt{\frac{1-cosx}{2}}\Rightarrow\sin^2\frac{x}{2}=\frac{1-cosx}{2}\Rightarrow 2\sin^2\frac{x}{2}=1-cosx$

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}=\lim_{x\to 0}\frac{2\sin x-2\sin x\cos x}{x^3}=\lim_{x\to 0}\frac{2\sin x(1-\cos x)}{x^3}=$$

12. $$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{2(1-\cos x)}{x^2}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{2(2\sin^2\frac{x}{2})}{x^2}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{4\left(\sin^2\frac{x}{2}\right)\cdot\frac{1}{4}}{x^2\cdot\frac{1}{4}}=$$

$$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\left(\sin^2\frac{x}{2}\right)}{x^2\cdot\frac{1}{4}}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}=1\cdot 1\cdot 1=1$$

Identity : $\sin(-x)=-\sin x\Rightarrow\sin(-2x^2)=-2\sin(x^2)$

13. $$\lim_{x\to 0}\frac{\sin(-2x^2)}{x^2}=\lim_{x\to 0}\frac{-2\cdot\sin(x^2)}{x^2}=-2\cdot\lim_{x\to 0}\frac{\sin(x^2)}{x^2}=-2\cdot 1=-1$$

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 4\left(x-\frac{\pi}{2}\right)}{2\left(x-\frac{\pi}{2}\right)}=\frac{4}{2}\cdot\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}=2\cdot\lim_{x\to\frac{\pi}{2}}\frac{\frac{\sin\left(x-\frac{\pi}{2}\right)}{\cos\left(x-\frac{\pi}{2}\right)}}{\left(x-\frac{\pi}{2}\right)}=$$

14. $$2\cdot\lim_{x\to\frac{\pi}{2}}\frac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)\cdot\cos\left(x-\frac{\pi}{2}\right)}=2\cdot\lim_{x\to\frac{\pi}{2}}\frac{\sin\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)}\cdot\frac{1}{\cos\left(x-\frac{\pi}{2}\right)}=$$

$$2\cdot 1\cdot\frac{1}{\cos\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}=2\cdot 1\cdot\frac{1}{\cos(0)}=2\cdot 1\cdot 1=2$$

15. $$\lim_{x \to 1} \frac{\sin^2(x-1)}{(x-1)} = \lim_{x \to 1} \frac{\sin(x-1)\sin(x-1)}{(x-1)} = 1 \cdot \sin(1-1) = 1 \cdot \sin 0 = 1 \cdot 0 = 0$$

16. $$\lim_{x \to 0} \frac{3+\tan x}{x} = \lim_{x \to 0} \frac{3+\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{3}{x} + \frac{\sin x}{x\cos x} =$$

$$\lim_{x \to 0} \frac{3}{x} + \frac{\sin x}{x} \cdot \frac{1}{\cos x} = \frac{3}{0} + 1 \cdot 1$$ no limit exists