# Functions

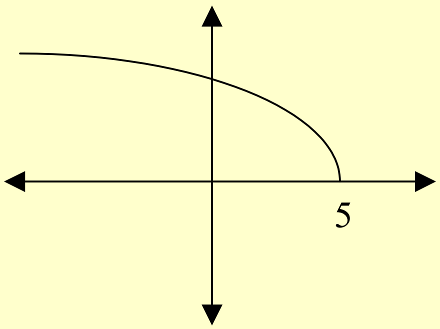
A quantity $y$ is a function of another quantity $x$ if there is some rule (an algebraic equation, a graph, a table, or as an English description) by which a unique value is assigned to $y$ by a corresponding value of $x$. The rule can never be ambiguous. A **function** is a set of ordered pairs of numbers $(x, y)$, in which no two distinct ordered pairs have the same first number. This restriction assures that $y$ is unique for a specific value of $x$.

Since values are assigned to $x$ and the value of $y$ is dependent upon the choice of $x$, $x$ is referred to as the **independent** variable and $y$ the **dependent** variable. The **domain** of a function is the totality of all possible values of the independent variable, and the **range** of the function is the totality of all possible values of the dependent variable.

If the ordered pairs of numbers $(x, y)$ for a specific function are plotted as cartesian coordinates of a point on a plane, the totality of those points is referred to as the **graph of the function**. Since for each value of $x$ in the domain of the function there corresponds a unique value of $y$. No vertical line line can intersect the graph of the function in more than one point (**The Vertical Line Test**).

## Examples:

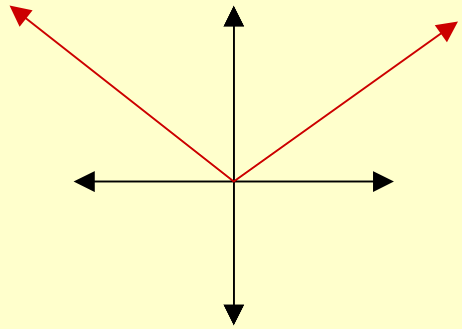
$y = \sqrt{5 - x}$




domain : $(-\infty, 5]$

range : $[0, +\infty)$ *or* $[0, \infty)$

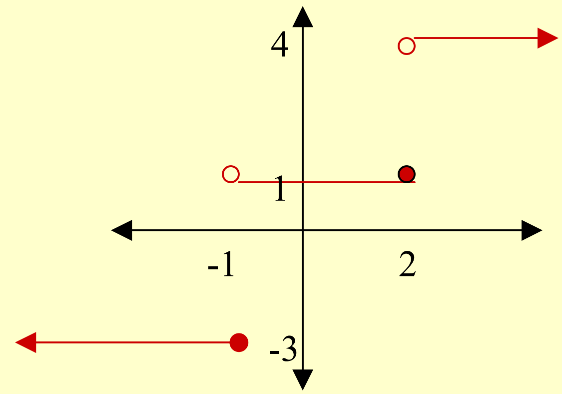
$y = |x|$



domain : $(-\infty, \infty)$

range : $[0, \infty)$

$$y = \begin{cases} -3 & \text{if } x \le -1 \\ 1 & \text{if } -1 < x \le 2 \\ 4 & \text{if } 2 < x \end{cases}$$



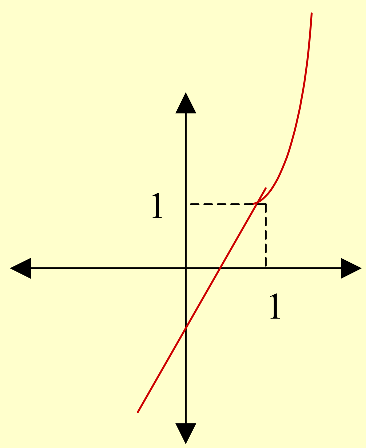

domain : $(-\infty, \infty)$

range : $\{-3, 1, 4\}$

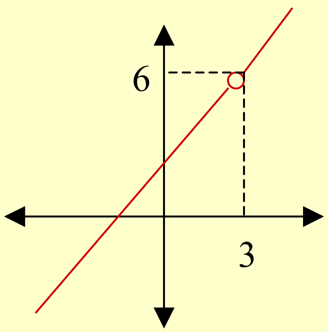
$$y = \begin{cases} 3x - 2 & \text{if } x < 1 \\ x^2 & \text{if } 1 \le x \end{cases}$$

domain : $(-\infty, \infty)$

range : $(-\infty, \infty)$



$$y = \frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{(x-3)}$$

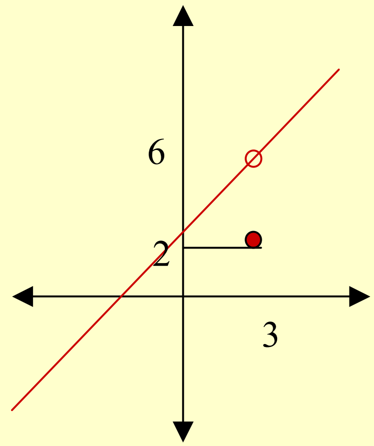



domain: $(-\infty, 3) \cup (3, \infty)$

range: $(-\infty, 6) \cup (6, \infty)$

When x = 3, both numerator and denominator are zero, and 0/0 is undefined. When factored, y = x + 3 and since x cannot equal 3, the range is all values except 6.

$$y = \begin{cases} x + 3 & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$$

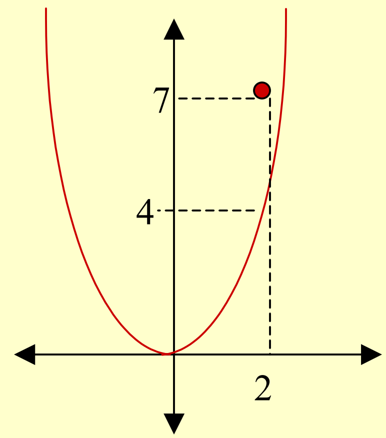



domain $(-\infty, 3) \cup (3, \infty)$

range: $(-\infty, 6) \cup (6, \infty)$

Graph consists of the point (3, 2) and all points on the line y = x + 3, except the point (3, 6)

$$y = \begin{cases} x^2 & \text{if } x \neq 2 \\ 7 & \text{if } x = 2 \end{cases}$$




*domain*: $(-\infty, \infty)$

*range*: $[0, \infty)$

Graph consists of the point (2, 7) and all the points on the parabola except (2, 4)

# Function Notation and Operations on Functions

If $f$ is the function having as its domain the values of $x$ and as its range the values of $y$, we use the symbol $f(x)$ (read " f of x") to denote the particular value of $y$ that corresponds to the value of $x$. Other symbols can $g(x)$ , $h(x)$ , $d(x)$ etc. When defining a function, the domain of the independent variable must be given, either stated or implied.

Examples:

$f(x) = 3x^2 - x + 2$ - implies "x" may be any real number.

$f(x) = 3x^2 - x + 2, \quad -1 \leq x < 3$ - stated that domain consists of all real numbers greater than or equal to -1 and less than 3.

$g(x) = \dfrac{5}{x^2 - 9}$ - implies that x cannot equal 3 or -3, hence the domain is all real numbers except -3 and 3.

$h(x) = \sqrt{16 - x^2}$ - implies that x exists in the closed interval from -4 to 4, hence the domain is [-4, 4].

Given the two function ***f*** and ***g*** :

1. Their sum, denoted by ***(f + g)*** , is the function defined by ***(f + g)x= f(x) + g(x)***
2. Their difference, denoted by ***(f - g)*** , is the function defined by ***(f - g)x= f(x) - g(x)***
3. Their product, denoted by ***(f g)*** , is the function defined by ***(f g)x= f(x) g(x)***
4. Their quotient, denoted by ***(f / g)*** , is the function defined by ***(f / g)x= f(x) / g(x)***
5. The product of a function f multiplied by itself ***( f f)*** is defined by $(f(x))^2$

**Even** and **Odd** Functions .

A function $f$ is even if $f(-x) = f(x)$ for all x in its domain;
A function $f$ is odd if $f(-x) = -f(x)$ for all x in its domain

**Periodic Functions**.

Any function is called periodic if it "repeats" itself on intervals of any fixed length. For example the sine curve. Periodicity may be defined symbolically:

***A function f is periodic with period P if the equation f (x+p) = f(x) holds for all x in the domain of f.***

The definition says that the graph of $f$ repeats itself on intervals of length P. For any function $f$, the graph of $y = f(x + P)$ is the result of shifting the graph of $y = f(x)$ a distance of P units. In simple words, the graph of the function does not change when it is shifted P units.

**A Composite Function:**

Let $f$ and $g$ be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the composite of $f$ with $g$. The domain of $f \circ g$ is the set of all $x$ in the domain such that $g(x)$ is in the domain of $f$.

**Basic Types of Transformations:**

| | | | |
|---|---|---|---|
| original graph: | $y = f(x)$ | vertical shift c units upwards: | $y = f(x) + c$ |
| horizontal shift c units to the right: | $y = f(x - c)$ | reflection about the x-axis: | $y = -f(x)$ |
| horizontal shift c units to the left: | $y = f(x + c)$ | reflection about the y-axis: | $y = f(-x)$ |
| vertical shift c units downwards: | $y = f(x) - c$ | | |

# ELEMENTARY FUNCTIONS

An **elementary function** is one built from certain legal basic elements (powers of a variable, a trig function, a log function, etc.) using certain legal operations ( addition, subtraction, etc.). For example: $f(x) = \dfrac{\ln(\sin(2x))}{1+3x}$

**Function families**. These include algebraic and transcendental functions.

1. **Algebraic Functions** are defined using only the ordinary algebraic operations: addition, subtraction, division, multiplication, raising to powers and taking roots.

   a) **Polynomials**. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,\ a_n \neq 0$

   where the positive integer $n$ is the degree of the polynomial function and the numbers $a_i$ are coefficients, with $a_n$ the leading coefficient and $a_0$ the constant term. Polynomials are the simplest algebraic function; their rules require only multiplication and addition. The polynomial expression $2x^3 + 5x$ corresponds naturally to the function $p(x) = 2x^3 + 5x$. Any function defined by a **polynomial expression** is called a **polynomial function**. Because polynomials involve only multiplication and addition they accept all real numbers as inputs: that is, every polynomial function has the same natural domain--all real numbers. The range for each function is defined by each particular polynomial function and graphic analysis can be of great benefit. Polynomial graphs are everywhere continuous and smooth - they have neither breaks or kinks.

Types:

a) The simplest polynomials are constants.

b) The simplest interesting polynomials are **linear function** (first power of x) and each graph is that of a straight line.

c) **Quadratic and cubic polynomials** have their characteristic parabolic and cubic curves.

Polynomials in general can take almost any smooth, unbroken shape. Some shape restrictions do exist. A polynomial of degree "n" can have at most n roots; this means, graphically, that a polynomial graph can have at most "n" x-intercepts.

b) **Rational Functions**. If a function can be expressed as the quotient of two polynomial functions, the function is called a rational function.Examples:

$$f(x) = \frac{1}{x}, g(x) = \frac{x^2}{x^2+3}, h(x) = \frac{2}{x} + \frac{3}{x+1}$$

Rational graphs have an important new feature: the possibility of horizontal and vertical asymptotes. **Asymptotes** are straight lines toward which a graph "tends" but never touches or crosses. **Vertical asymptotes** correspond to the real roots of the denominator. The **horizontal asymptotes** reflect the functions "long run" behaviour. It shows how the function behaves for large positive and large negative inputs.

2. **Transcendental Functions**. These include: trigonometric functions, inverse trigonometric functions, logarithmic functions and exponential functions.

a) **Trigonometric Functions**. In this course the trig functions normally associated with right triangles will be functions in their own right. The most important property of trig functions is their **repetitive or periodic behaviour**. For calculus purposes it's best to think of trig functions in terms of circles.

*Definition: For any real number x, let P(x) be the point reached by moving x units of distance counterclockwise around the unit circle. starting from (1, 0). (If x < 0, go clockwise.) Then*
*cos(x) = u - coordinate of P(x) and sin(x) = v - coordinate of P(x)*

The domain for the sin and cos curves allows for every real number "x" to be a legal input; that is (- ∞ , ∞). For the other trig functions the domain is also defined for all real numbers except those at which a denominator is zero. The range for the sin and cos curves is always between [-1, 1]; while the range for the remaining trig function is defined as (- ∞, ∞ ). The sin and tan curves with their respective reciprocals are odd functions (i.e. sin (-x) = - sin (x)), while the cos and sec curves are even (i.e. cos (-x) = cos (x)). The tan, cot, sec, and csc functions all have vertical asymptotes. Of special note is the geometric interpretation of tan, the unit circle and slope.

$$\tan x = \frac{\sin x}{\cos x} = \text{slope of the line from the origin to } P(x)$$

b) An **exponential function** is defined by the expression of the form $f(x) = b^n$ where "b" is a fixed positive number, called the base. Each of the expressions

$$2^x,\ 4^t,\ \left(\frac{1}{2}\right)^z, and\ \ e^x$$

defines an exponential function. Each of the following

$$x^2, z^3,\ and\ (w+1)^{15}$$

do not define an exponential function because each involves a fixed power of a variable. Of all exponential functions, the one with base "***e*** " turns out to be very useful. The number "***e*** " is irrational (***e*** = 2.718328182845904....). The function $f(x) = b^n$ is often called the exponential function or natural exponential function. Alternative notations include: exp(x) = y, exp x = y, and $e^x = y$ and are read as "the value of the exponential function at x". Exponential function accept all real inputs so all have a domain of (- ∞ , ∞ ). Graphs indicate that unless b = 1, the range of the exponential function $f(x) = b^n$ is (0, ∞ ). The shape of the exponential graph depends on base "b". The larger the value of "b", the faster $b^x$ increases. Every graph of the form $f(x) = b^n$ passes through the point (0, 1). Exponential functions are **monotone**; that is "they are everywhere increasing or everywhere decreasing"

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c) A **logarithm function** is defined by an expression of the form $f(x) = \log_b x$

Each of the expressions $\log_2 x, \quad \log_4 x, \quad \log_e x$

defines a log function with some base "***b*** ". The log function with base "***e*** " is called the natural log function and is denoted by **ln(x) or ln x**. The log function accepts positive inputs and outputs take all real values. Thus the domain for a log function is (0, ∞) and the range is (- ∞ , ∞ ). Each graphs shape depends on the value of "b". These graphs rise more and more slowly as x increases. The graph of every log function passes through the point (1, 0). Log functions are **monotone.** If b > 1, the graph always rises and if 0 < b < 1, the graph always falls.

Logarithm and exponential function with base "b" are inverses of each other. Specifically, each log graph is the reflection of the corresponding exponential graph across the line y = x

# Special Functions

## 1. Postage Function

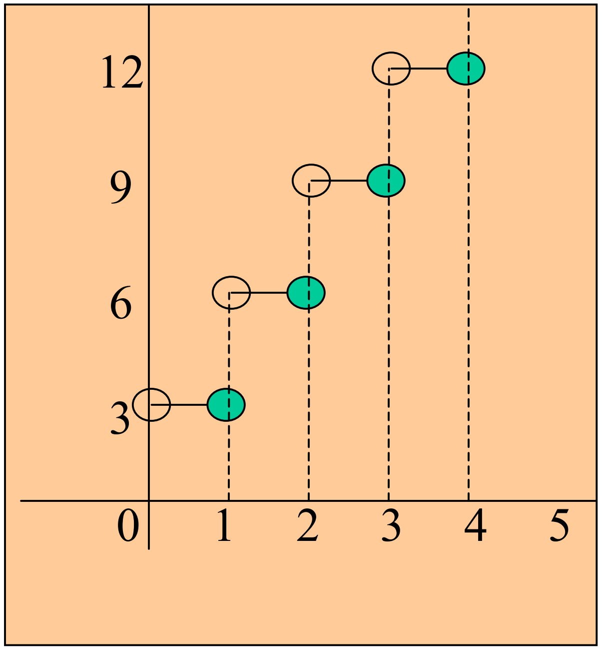
$f(x) = 3n \quad if \quad n-1 < x \le n$

$y = 3 \quad if \; 0 < x \le 1$

$y = 6 \quad if \; 1 < x \le 2$

$y = 9 \quad if \; 2 < x \le 3$

$y = 12 \quad if \; 3 < x \le 4$




## 2. Greatest Integer Function

$f(x) = [[x]]$

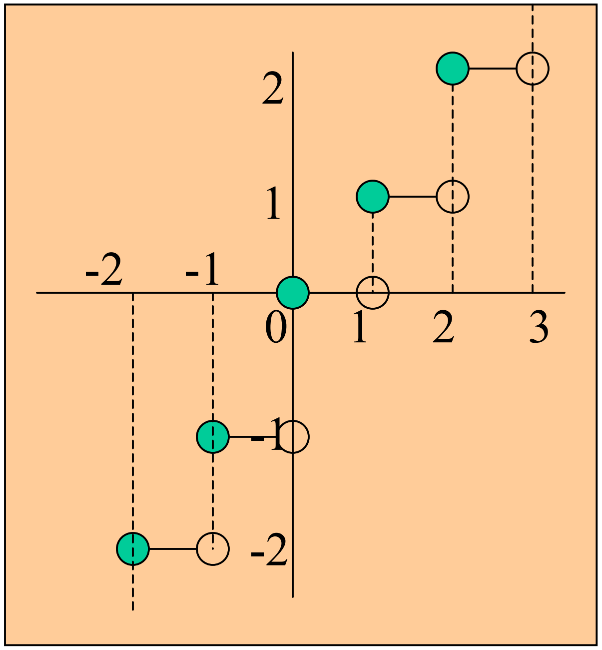
$-2 \le x < 1 \quad [[x]] = -2$

$-1 \le x < 0 \quad [[x]] = -1$

$0 \le x < 1 \quad [[x]] = 0$

$1 \le x < 2 \quad [[x]] = 1$

$2 \le x < 3 \quad [[x]] = 2$




## 3. Unit Step Function

$$U(t) = \begin{cases} 0 & if \; t < 0 \\ 1 & if \; t \ge 0 \end{cases}$$