Implicit Differentiation

a) With respect to x

$$x^2y + xy^2 = 6$$

1. $2x\cdot y + 1\cdot\frac{dy}{dx}\cdot x^2 + 1\cdot y^2 + 2y\frac{dy}{dx}\cdot x = 0 \Rightarrow \frac{dy}{dx}\left(x^2 + 2xy\right) = -2xy - y^2$

$$\frac{dy}{dx} = \frac{\left(-2xy - y^2\right)}{\left(x^2 + 2xy\right)} = \frac{-y(2x+y)}{x(x+2y)}$$

$$x^3 - xy + y^3 = 1$$

2. $3x^2 - \left(1\cdot y + 1\cdot\frac{dy}{dx}\cdot x\right) + 3y^2\frac{dy}{dx} = 0 \Rightarrow 3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$$\frac{dy}{dx}\left(-x + 3y^2\right) = -3x^2 + y \Rightarrow \frac{dy}{dx} = \frac{\left(-3x^2 + y\right)}{\left(-x + 3y^2\right)}$$

$$2xy + y^2 = x + y$$

3. $2\cdot y + 1\cdot\frac{dy}{dx}\cdot 2x + 2y\frac{dy}{dx} = 1 + 1\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left(2x + 2y - 1\right) = -2y + 1 \Rightarrow$

$$\frac{dy}{dx} = \frac{\left(-2y + 1\right)}{\left(2x + 2y - 1\right)}$$

$$x^2(x-y)^2 = x^2 - y^2 \Rightarrow x^2(x^2 - 2xy + y^2) = x^2 - y^2 \Rightarrow x^4 - 2x^3y + x^2y^2 = x^2 - y^2$$

$$4x^3 + \left(-2\cdot 3x^2\cdot y + 1\cdot\frac{dy}{dx}\cdot -2x^3\right) + \left(2x\cdot y^2 + 2y\frac{dy}{dx}\cdot x^2\right) = 2x - 2y\frac{dy}{dx} \Rightarrow$$

4. $4x^3 - 6x^2y - 2x^3\frac{dy}{dx} + 2xy^2 + 2x^2y\frac{dy}{dx} = 2x - 2y\frac{dy}{dx} \Rightarrow$

$$\frac{dy}{dx}\left(-2x^3 + 2x^2y + 2y\right) = -4x^3 + 6x^2y + 2x \Rightarrow$$

$$\frac{dy}{dx} = \frac{\left(-4x^3 + 6x^2y + 2x\right)}{\left(-2x^3 + 2x^2y + 2y\right)} \Rightarrow \frac{dy}{dx} = \frac{-2\left(2x^3 - 3x^2y - x\right)}{-2\left(x^3 - x^2y - y\right)} \Rightarrow \frac{dy}{dx} = \frac{\left(2x^3 - 3x^2y - x\right)}{\left(x^3 - x^2y - y\right)}$$

5.

$$y^2 = \frac{x-1}{x+1} \Rightarrow y^2 = (x-1)(x+1)^{-1}$$

$$2y\frac{dy}{dx} = 1\cdot(x+1)^{-1} + -1\cdot(x+1)^{-2}\cdot 1\cdot(x-1) \Rightarrow 2y\frac{dy}{dx} = (x+1)^{-2}\big((x+1)-(x-1)\big) \Rightarrow$$

$$2y\frac{dy}{dx} = (x+1)^{-2}\left(x+1-x+1\right) \Rightarrow 2y\frac{dy}{dx} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{2}{(x+1)^2\cdot 2y} = \frac{1}{y(x+1)^2}$$

6.

$$x^2 = \frac{x-y}{x+y} \Rightarrow x^2 = \left(x-y\right)\left(x+y\right)^{-1}$$

$$2x = \left(1-\frac{dy}{dx}\right)\left(x+y\right)^{-1} + -1\cdot\left(x+y\right)^{-2}\left(1+\frac{dy}{dx}\right)\left(x-y\right) \Rightarrow$$

$$2x = \left(x+y\right)^{-2}\left(\left(1-\frac{dy}{dx}\right)\left(x+y\right) - \left(1+\frac{dy}{dx}\right)\left(x-y\right)\right) \Rightarrow$$

$$2x = \left(x+y\right)^{-2}\left(x+y-x\frac{dy}{dx}-y\frac{dy}{dx}-\left[x-y+x\frac{dy}{dx}-y\frac{dy}{dx}\right]\right) \Rightarrow$$

$$2x = \left(x+y\right)^{-2}\left(x+y-x\frac{dy}{dx}-y\frac{dy}{dx}-x+y-x\frac{dy}{dx}+y\frac{dy}{dx}\right) \Rightarrow 2x = \left(x+y\right)^{-2}\left(2y-2x\frac{dy}{dx}\right) \Rightarrow$$

$$2x = \frac{1}{\left(x+y\right)^2}\left(2y-2x\frac{dy}{dx}\right) \Rightarrow 2x\left(x+y\right)^2 = 2y-2x\frac{dy}{dx} \Rightarrow 2x\left(x+y\right)^2 - 2y = -2x\frac{dy}{dx} \Rightarrow$$

$$\frac{2x\left(x+y\right)^2-2y}{-2x} = \frac{dy}{dx}$$

7.

$$x = \tan y \Rightarrow x = \frac{\sin y}{\cos y} \Rightarrow x = \sin y\left(\cos y\right)^{-1}$$

$$1 = \cos y\frac{dy}{dx}\cdot\left(\cos y\right)^{-1} + -1\left(\cos y\right)^{-2}\cdot -\sin y\frac{dy}{dx}\cdot\sin y \Rightarrow$$

$$1 = \frac{dy}{dx} + \frac{\sin^2 y}{\cos^2 y}\frac{dy}{dx} \Rightarrow 1 = \left(1+\frac{\sin^2 y}{\cos^2 y}\right)\frac{dy}{dx} \Rightarrow 1 = \left(\frac{\cos^2 y+\sin^2 y}{\cos^2 y}\right)\frac{dy}{dx} \Rightarrow$$

$$1 = \left(\frac{1}{\cos^2 y}\right)\frac{dy}{dx} \Rightarrow \cos^2 y = \frac{dy}{dx}$$

or

$$1 = \frac{dy}{dx} + \frac{\sin^2 y}{\cos^2 y}\frac{dy}{dx} \Rightarrow 1 = \left(1+\tan^2 y\right)\frac{dy}{dx} \Rightarrow \frac{1}{\left(1+\tan^2 y\right)} = \frac{dy}{dx} \Rightarrow \frac{1}{\sec^2 y} = \cos^2 y = \frac{dy}{dx}$$

8. $x + \sin y = xy$

$$1 + \cos y\frac{dy}{dx} = 1 \cdot y + 1 \cdot \frac{dy}{dx} \cdot x \Rightarrow \frac{dy}{dx}\left(\cos y - x\right) = -1 + y \Rightarrow \frac{dy}{dx} = \frac{\left(-1 + y\right)}{\left(\cos y - x\right)}$$

b) Find the slope at the given point

1. $y^2 + x^2 = y^4 - 2x$ at $(-2,-1)$

$$2y\frac{dy}{dx} + 2x = 4y^3\frac{dy}{dx} - 2 \Rightarrow \frac{dy}{dx}\left(2y - 4y^3\right) = -2x - 2 \Rightarrow \frac{dy}{dx} = \frac{\left(-2x - 2\right)}{\left(2y - 4y^3\right)}$$

$$\frac{dy}{dx} = \frac{\left(-2(-2) - 2\right)}{\left(2(-1) - 4(-1)^3\right)} = \frac{4 - 2}{-2 + 4} = \frac{2}{2} = 1$$

2.

$\left(x^2 + y^2\right)^2 = \left(x - y\right)^2$ at $(1,-1)$

$$x^4 + 2x^2y^2 + y^4 = x^2 - 2xy + y^2$$

$$4x^3 + 4x \cdot y^2 + 2y\frac{dy}{dx} \cdot 2x^2 + 4y^3\frac{dy}{dx} = 2x + \left(-2 \cdot y + 1 \cdot \frac{dy}{dx} \cdot -2x\right) + 2y\frac{dy}{dx}$$

$$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 2x + -2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(4x^2y + 4y^3 + 2x - 2y\right) = -4x^3 - 4xy^2 + 2x + -2y$$

$$\frac{dy}{dx} = \frac{\left(-4x^3 - 4xy^2 + 2x + -2y\right)}{\left(4x^2y + 4y^3 + 2x - 2y\right)} = \frac{\left[-4(1)^3 - 4(1)(-1)^2 + 2(1) + -2(-1)\right]}{\left[4(1)^2(-1) + 4(-1)^3 + 2(1) - 2(-1)\right]} = \frac{\left[-4 - 4 + 2 + 2\right]}{\left[-4 - 4 + 2 + 2\right]} = \frac{-4}{-4} = 1$$

c) Verify that the point is on the curve and find the equations of the lines that are a) tangent and b) normal (a line perpendicular to a tangent line at the point of tangency) to the original curve.

1. Point Verification

$x^2 + xy - y^2 = 1, \quad (2,3) \Rightarrow \left[(2)^2 + (2)(3) - (3)^2\right] = 1 \Rightarrow \left[4+6-9\right] = 1 \Rightarrow \left[1\right] = 1$

$2x + 1\cdot y + 1\cdot\frac{dy}{dx}\cdot x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}\left(x-2y\right) = -2x - y \Rightarrow \frac{dy}{dx} = \frac{\left(-2x-y\right)}{\left(x-2y\right)} \Rightarrow$

$\frac{dy}{dx} = \frac{\left[-2x-y\right]}{\left[x-2y\right]} = \frac{\left[-2(2)-(3)\right]}{\left[(2)-2(3)\right]} = \frac{\left[-4-3\right]}{\left[2-6\right]} = \frac{-7}{-4} = \frac{7}{4}$

tangent equation

slope = $\frac{7}{4}$, point =(2,3)

$(y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y-3) = \frac{7}{4}(x-2) \Rightarrow 4y - 12 = 7x - 14 \Rightarrow 4y = 7x - 2$

normal equation - remember we need the negative reciprocal of the slope of $\perp$ lines

slope = $-\frac{4}{7}$, point =(2,3)

$(y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y-3) = \frac{-4}{7}(x-2) \Rightarrow 7y - 21 = -4x + 8 \Rightarrow 7y = -4x + 29$

2. Point Verification

$x^2y^2 = 9, \quad (-1,3) \Rightarrow \left[(-1)^2(3)^2\right] = 9 \Rightarrow \left[9\right] = 9$

$2x\cdot y^2 + 2y\frac{dy}{dx}\cdot x^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{-2xy^2}{2x^2y} = \frac{-y}{x} \Rightarrow \frac{dy}{dx} = \frac{-(3)}{-1} = 3$

Tangent equation

$m = 3, pt. = (-1,3)$

$(y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y-3) = 3(x-(-1)) \Rightarrow y - 3 = 3x + 3 \Rightarrow y = 3x + 6$

Normal equation

$m = \frac{-1}{3}, pt. = (-1,3)$

$(y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y-3) = \frac{-1}{3}(x-(-1)) \Rightarrow 3y - 9 = -x - 1 \Rightarrow 3y = -x + 8$

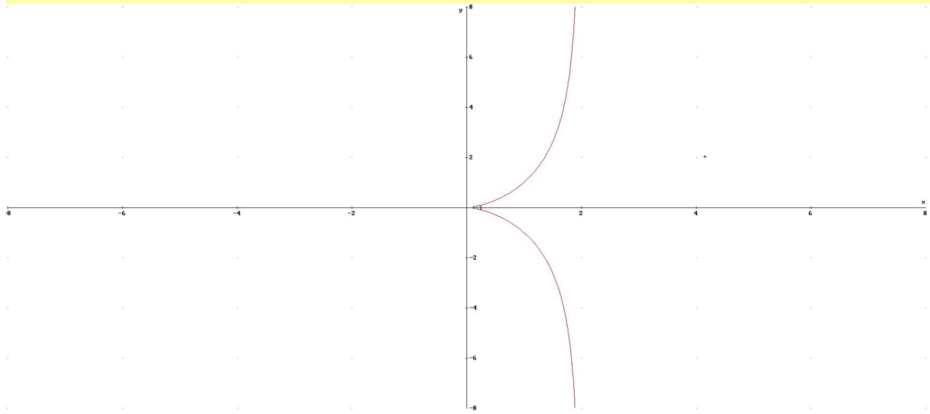
d) Sketch each of the equations and determine the slope at the given point of the following:

1. the cissoid of Diocies -

$$y^2(2-x) = x^3 \text{ at } (1,1) \Rightarrow 2y^2 - xy^2 = x^3$$

$$4y\frac{dy}{dx} - 1\cdot y^2 + 2y\cdot\frac{dy}{dx}\cdot -x = 3x^2 \Rightarrow \frac{dy}{dx}\left(4y-2xy\right) = y^2+3x^2 \Rightarrow \frac{dy}{dx} = \frac{\left(y^2+3x^2\right)}{\left(4y-2xy\right)}$$

$$\frac{dy}{dx} = \frac{\left[(1)^2+3(1)^2\right]}{\left[4(1)-2(1)(1)\right]} = \frac{\left[1+3\right]}{\left[4-2\right]} = \frac{\left[4\right]}{\left[2\right]} = 2$$



2.the devil's curve -

$$y^4 - 4y^2 = x^4 - 9x^2 \quad \text{at } (-3,2)$$

$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 18x \Rightarrow \frac{dy}{dx}\left(4y^3-8y\right) = \left(4x^3-18x\right) \Rightarrow \frac{dy}{dx} = \frac{\left(4x^3-18x\right)}{\left(4y^3-8y\right)}$$

$$\frac{dy}{dx} = \frac{\left[4(-3)^3-18(-3)\right]}{\left[4(2)^3-8(2)\right]} = \frac{\left[-108+54\right]}{\left[32-16\right]} = \frac{\left[-54\right]}{\left[16\right]} = -27/8$$