Higher Order Derivatives, Implicit Differentiation

1. Find the 1st and 2nd order derivatives for:

$$f(x) = 5x^4 + 2x^3 - 7x \Rightarrow$$

a) $f'(x) = 20x^3 + 6x^2 - 7 \Rightarrow$

$$f''(x) = 60x^2 + 12x$$

$$f(x) = \cos^3(5x) \Rightarrow$$

b)

$$f'(x) = 3\cos^2(5x) \cdot -\sin(5x) \cdot 5 = -15\cos^2(5x) \cdot \sin(5x) \Rightarrow$$

$$f''(x) = -15 \cdot 2 \cdot \cos(5x) \cdot -\sin(5x) \cdot 5 \cdot \sin(5x) + \cos(5x) \cdot 5 \cdot -15\cos^2(5x) =$$

$$75 \cdot \cos(5x)\left[2 \cdot \sin^2(5x) - \cos^2(5x)\right]$$

$$f(x) = e^{2x}\ln x \Rightarrow$$

$$f'(x) = e^{2x} \cdot 2 \cdot \ln x + \frac{1}{x}e^{2x} = e^{2x}\left(2\ln x + \frac{1}{x}\right) = e^{2x}\left(2\ln x + x^{-1}\right)$$

$$f''(x) = e^{2x} \cdot 2 \cdot \left(2\ln x + x^{-1}\right) + \left(2 \cdot \frac{1}{x} - 1 \cdot x^{-2}\right)e^{2x} = e^{2x}\left[2\left(2\ln x + x^{-1}\right) + \left(\frac{2}{x} - x^{-2}\right)\right] =$$

c)

$$e^{2x}\left[2\left(2\ln x + x^{-1}\right) + \left(\frac{2}{x} - \frac{1}{x^2}\right)\right] = e^{2x}\left[2\left(2\ln x + x^{-1}\right) + \left(\frac{2x-1}{x^2}\right)\right] =$$

$$e^{2x}\left[\frac{2x^2\left(2\ln x + x^{-1}\right) + (2x-1)}{x^2}\right] = e^{2x}\left[\frac{2x^2\left(2\ln x + \frac{1}{x}\right) + (2x-1)}{x^2}\right] =$$

$$e^{2x}\left[\frac{2x^2\left(\frac{2x\ln x + 1}{x}\right) + (2x-1)}{x^2}\right] = e^{2x}\left[\frac{4x^2\ln x + 4x - 1}{x^2}\right]$$

2. If a particle is projected vertically upward from ground level with an initial velocity v_o, its height after "t" seconds is $s(t) = v_o t - 16t^2 meters$. Suppose $v_o = 800$ meters per second.
   a) What is the velocity of the particle at time "t"?
      $s(t) = 800t - 16t^2 \Rightarrow s'(t) = 800 - 32t$

   b) At what time does the particle reach its maximum height?
      Max. height occurs when velocity (slope) is zero
      $0 = 800 - 32t \Rightarrow -800 = -32t \Rightarrow 25 = t$

   c) What is the maximum height?
      $s(t) = 800t - 16t^2 \Rightarrow 800(25) - 16(25)^2 = 10{,}000$

d) How long doe it take to reach the ground?
It will take 50 seconds. Half the time going up (25 sec) and the same time coming down (25 sec),

e) At what time(s) is the object at a height of 9216 feet?
$s(t) = 800t - 16t^2 \Rightarrow 9216 = 800t - 16t^2 \Rightarrow 16t^2 - 800t + 9216 = 0 \Rightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-800) \pm \sqrt{(-800)^2 - 4(16)(9216)}}{2(16)} = \frac{800 \pm 224}{32} = 18 \text{ or } 32$$

f) What is the velocity when it is at a height of 9216 feet?
$velcoity = s'(t) = 800 - 32t$

at time 18 seconds $s'(t) = 800 - 32t \Rightarrow 800 - 32(18) = 224 m/s$

at time 32 seconds $s'(t) = 800 - 32t \Rightarrow 800 - 32(32) = -224 m/s$

g) Is the acceleration of the object constant?
Yes because the second derivative yields a value of -32.

3. Using Implicit Differentiation

$x^3 + x^2y + xy^2 + y^3 = 0 \text{(with respect to x)} \Rightarrow$

$$3x^2 + \left(2xy + 1 \cdot \frac{dy}{dx} \cdot x^2\right) + \left(1 \cdot y^2 + 2y \cdot \frac{dy}{dx} \cdot x\right) + 3y^2 \cdot \frac{dy}{dx} = 0 \Rightarrow$$

a) $3x^2 + 2xy + x^2 \cdot \frac{dy}{dx} + y^2 + 2xy \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} = 0 \Rightarrow$

$$\frac{dy}{dx}\left(x^2 + 2xy + 3y^2\right) = -3x^2 - y^2 \Rightarrow$$

$$\frac{dy}{dx} = \frac{-3x^2 - y^2}{x^2 + 2xy + 3y^2}$$

$x \sin y = y \cos x \text{(with respect to x)} \Rightarrow$

$$\sin y + x \cos y \cdot \frac{dy}{dx} = 1 \cdot \frac{dy}{dx} \cdot \cos x + (-\sin x) \cdot y \Rightarrow$$

b) $x \cos y \cdot \frac{dy}{dx} - \cos x \cdot \frac{dy}{dx} = -\sin y - y \sin x \Rightarrow$

$$\frac{dy}{dx}\left(x \cos y - \cos x\right) = -\sin y - y \sin x \Rightarrow$$

$$\frac{dy}{dx} = \frac{-\sin y - y \sin x}{x \cos y - \cos x}$$

c) $3x^4 - 4y^3 = 12$ find the second order derivative with respect to x $\Rightarrow$

first order $12x^3 - 12y^2 \dfrac{dy}{dx} = 0 \Rightarrow -12y^2 \dfrac{dy}{dx} = -12x^3 \Rightarrow \dfrac{dy}{dx} = \dfrac{-12x^3}{-12y^2} = \dfrac{x^3}{y^2} = x^3y^{-2}$

second order $\dfrac{d^2y}{d^2x} = 3x^2y^{-2} + -2y^{-3}\dfrac{dy}{dx} \cdot x^3 \Rightarrow \dfrac{d^2y}{d^2x} = 3x^2y^{-2} + -2y^{-3}\left(x^3y^{-2}\right) \cdot x^3 \Rightarrow$

$\dfrac{d^2y}{d^2x} = 3x^2y^{-2} - 2x^6y^{-5} = x^2y^{-5}\left(3y^3 - 2x^4\right) = \dfrac{x^2\left(3y^3 - 2x^4\right)}{y^5}$

4. Determine the equation of a tangent line to the curve $x^2 + 4y^3 = -28$ at y = -2

$$x^2 + 4y^3 = -28 \Rightarrow 2x + 12y^2 \frac{dy}{dx} = 0 \Rightarrow 12y^2 \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{12y^2} = \frac{-x}{6y^2}$$

Point of tangency = $x^2 + 4y^3 = -28 \Rightarrow x^2 + 4(-2)^3 = -28 \Rightarrow x^2 - 32 = -28 \Rightarrow$
$x^2 = 4 \Rightarrow x = -2$ or $2 \Rightarrow$ Points $(-2,-2)$ and $(2,-2)$

slope(s) of tangent line = for the point (-2, -2) = $\dfrac{-x}{6y^2} = \dfrac{-2}{6(-2)^2} = \dfrac{-2}{24} = -\dfrac{1}{12}$

for the point (2, -2) = $\dfrac{-x}{6y^2} = \dfrac{2}{6(-2)^2} = \dfrac{2}{24} = \dfrac{1}{12}$

Equation(s) of the tangent lines :

a) $(y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y - (-2)) = -\dfrac{1}{12}(x - (-2)) \Rightarrow (y + 2) = -\dfrac{1}{12}(x + 2) \Rightarrow 12y + 24 = -x - 2 \Rightarrow$
$12y = -x - 26$

b) $(y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y - (-2)) = \dfrac{1}{12}(x - 2) \Rightarrow (y + 2) = \dfrac{1}{12}(x - 2) \Rightarrow 12y + 24 = x - 2 \Rightarrow$
$12y = x - 26$