Higher Order, Implicit Derivatives and Curve Sketching Answer key

A. Higher order derivatives: Determine the 1st and 2nd order derivatives for each of the following:

1. $f(x) = 5x^3 - 3x^5$

$f'(x) = 15x^2 - 15x^4 \Rightarrow 15x^2(1 - x^2)$

$f''(x) = 30x - 60x^3 \Rightarrow 30x(1 - 2x^2)$

2. $f(x) = \dfrac{x^3 + 7}{x}$

$f(x) = \dfrac{x^3 + 7}{x} \Rightarrow \dfrac{x^3}{x} + \dfrac{7}{x} \Rightarrow x^2 + 7x^{-1}$

$f'(x) = 2x + (-1)7x^{-2} \Rightarrow 2x - 7x^{-2}$

$f''(x) = 2 - (-2)7x^{-3} \Rightarrow 2 + 14x^{-3} \Rightarrow \dfrac{2x^3 + 14}{x^3} \Rightarrow \dfrac{2(x^3 + 7)}{x^3}$

or

$f(x) = \dfrac{x^3 + 7}{x} \Rightarrow (x^3 + 7)x^{-1}$

$f'(x) = 3x^2 \cdot x^{-1} + (-1)x^{-2}(x^3 + 7) \Rightarrow 3x - x^{-2}(x^3 + 7)$

$f'(x) = 3 - \left[(-2)x^{-3}(x^3 + 7) + 3x^2 \cdot x^{-2}\right] \Rightarrow 3 + 2x^{-3}(x^3 + 7) - 3 \Rightarrow \dfrac{2}{x^3}(x^3 + 7) \Rightarrow \dfrac{2(x^3 + 7)}{x^3}$

3. $f(x) = \sin(x^2)$

$f'(x) = \cos(x^2) \cdot 2x$

$f''(x) = \left[-\sin(x^2) \cdot 2x\right] \cdot 2x + 2\cos(x^2) \Rightarrow 2\left(-2x^2 \sin(x^2) + \cos(x^2)\right)$

B. Application:

A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec. It reaches a height of $s(t) = 160t - 16t^2$ feet after t sec.

a) How high does the rock go?

The instant that the rock reaches its highest point its velocity is zero. To find the max. height, determine the first derivative, equate to zero and find the times when the velocity is zero. Using this info to find s(t) - the rock's height.

velocity $\Rightarrow s'(t) = 160 - 32t$

time $\Rightarrow 160 - 32t = 0 \Rightarrow t = 5\text{sec}$

rock's height $\Rightarrow s(5) = 160(5) - 16(5^2) \Rightarrow s(5) = 800 - 400 = 400\,\textit{feet}$

b) What are the velocity and the speed of the rock when it is 256 feet above the ground on the way up? On the way down?

To find the rock's velocity at 256 feet on the way up and down, determine

the two values of t such that $s(t) = 160t - 16t^2 = 256$

times $\Rightarrow -16t^2 + 160t - 256 = 0 \Rightarrow -16(t-2)(t-8) = 0 \Rightarrow t = 2\,\text{sec}$ and $8\,\text{sec}$

using the info from part a)

$v(2) = s'(2) = 160 - 32(2) = 160 - 64 = 96\,ft/\sec$

$v(8) = s'(8) = 160 - 32(8) = 160 - 256 = -96\,ft/\sec$

c) What is the acceleration of the rock at any time t during its flight (after the blast)?

At any time during its flight following explosion, the rock's acceleration is a constant

acceleration $\Rightarrow s''(t) = -32\,ft/\sec$

it slows down, as it falls, it speeds up

d) When does the rock hit the ground?

The rock hits the ground at the positive time t for which s = 0. The equation $160t - 16t^2 = 0 \Rightarrow$

$-16t(10-t) = 0 \Rightarrow t = 0$ and $t = 10$. At t = 0,

returned to the ground 10 sec later.

C. Implicit Differentiation

1. $x^2y + xy^2 = 6$ with respect to x

$$\left(2x \cdot y + 1 \cdot \frac{dy}{dx} \cdot x^2\right) + \left(1 \cdot y^2 + 2y\frac{dy}{dx} \cdot x\right) = 0 \Rightarrow \frac{dy}{dx}\left(x^2 + 2xy\right) = -2xy - y^2 \Rightarrow \frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

2. $y^2 = x^2 + \sin xy$ with respect to x

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(1 \cdot y + x \cdot \frac{dy}{dx}\right) \Rightarrow 2y\frac{dy}{dx} = 2x + y\cos(xy) + x\cos(xy)\frac{dy}{dx} \Rightarrow$$

$$\frac{dy}{dx}\left(2y - x\cos(xy)\right) = 2x + y\cos(xy) \Rightarrow \frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

2. $2xy + y^2 = x + y$ with respect to y

$$\left(2\frac{dx}{dy} \cdot y + 2x\right) + 2y = 1 \cdot \frac{dx}{dy} + 1 \Rightarrow 2y\frac{dx}{dy} - \frac{dx}{dy} = -2y - 2x + 1 \Rightarrow \frac{dx}{dy}(2y - 1) = -2y - 2x + 1 \Rightarrow$$

$$\frac{dx}{dy} = \frac{-2y - 2x + 1}{2y - 1}$$

4. $2x^3 - 3y^2 = 8$ - find the 2nd derivative with respect to x

$$6x^2 - 6y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-6x^2}{-6y} = \frac{x^2}{y} = x^2y^{-1}$$

Substitution of first derivative

2nd derivative

$$\frac{d^2y}{dx^2} = 2xy^{-1} + (-1)y^{-2}\frac{dy}{dx} \cdot x^2 \Rightarrow xy^{-2}\left(2y - x\frac{dy}{dx}\right) \Rightarrow xy^{-2}\left(2y - xx^2y^{-1}\right) \Rightarrow$$

$$xy^{-3}\left(2y^2 - x^3\right) \Rightarrow \frac{x}{y^3}\left(2y^2 - x^3\right) \text{ or } \frac{2x}{y} - \frac{x^4}{y^3}$$

D. Application

Determine the equation of the line through the point (2, 3) tangent to the curve defined by the equation $x^2 + xy - y^2 = 1$

To find the slope, we need dy/dx

$x^2 + xy - y^2 = 1$

$2x + \left(1 \cdot y + 1 \cdot \frac{dy}{dx} \cdot x\right) - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x - 2y) = -2x - y \Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x - 2y}$

$m = \frac{dy}{dx} = \frac{-2x - y}{x - 2y} = \frac{-2(2) - 3}{2 - 2(3)} = \frac{-4 - 3}{2 - 6} = \frac{7}{4}$

$equation \Rightarrow (y_2 - y_1) = m(x_2 - x_1) \Rightarrow (y - 3) = \frac{7}{4}(x - 2) \Rightarrow 4y = 7x - 2$

E. Curve Sketching

Sketch the curve $f(x) = 4x^2(1 - x^2)$

a) x-intercepts $\Rightarrow 4x^2\left(1 - x^2\right) = 0 \Rightarrow x = 0, x = 1, x = -1$

b) y-intercepts $\Rightarrow y = 4(0)^2(1 - 0^2) = 0$

c) max/min and where graph is increasing/decreasing

$f'(x) = 8x(1 - x^2) + 4x^2(-2x) \Rightarrow 8x - 16x^3 \Rightarrow 8x(1 - 2x^2)$

critical numbers $x = 0, x = +\sqrt{2}/2, x = -\sqrt{2}/2$

maximum and minimums found by substituting into original equation

$f(x) = 4(0)^2(1 - 0^2) = 0, f(x) = 4\left(\sqrt{2}/2\right)^2\left(1 - \left(\sqrt{2}/2\right)^2\right) = 1, f(x) = 4\left(-\sqrt{2}/2\right)^2\left(1 - \left(-\sqrt{2}/2\right)^2\right) = 1$

max/min points (0, 0), $\left(\sqrt{2}/2, 1\right), \left(-\sqrt{2}/2, 1\right)$

Regions where graph increasing/decreasing

Interval	$\left(-\infty, -\sqrt{2}/2\right)$	$\left(-\sqrt{2}/2, 0\right)$	$\left(0, \sqrt{2}/2\right)$	$\left(\sqrt{2}/2, \infty\right)$
Value	-1	-½	½	1
Result	8	-2	2	-8
Sign	+	-	+	-
Interpret	increasing	decreasing	increasing	decreasing

d) points of inflection and regions of concavity

$f''(x) = 8(1-2x^2) + (-4x)\cdot(8x) = 8(1-6x^2)$

Critical numbers $\Rightarrow x = -\sqrt{6}/6, x = \sqrt{6}/6$

Point of inflection calculated by substituting into original equation

$f(x) = 4\left(-\sqrt{6}/6\right)^2\left(1-\left(-\sqrt{6}/6\right)^2\right) = 5/9, f(x) = 4\left(\sqrt{6}/6\right)^2\left(1-\left(\sqrt{6}/6\right)^2\right) = 5/9$

Points of inflection $\left(-\sqrt{6}/6, 5/9\right), \left(\sqrt{6}/6, 5/9\right)$

Regions of Concavity

Interval	$\left(-\infty, -\sqrt{6}/6\right)$	$\left(-\sqrt{6}/6, \sqrt{6}/6\right)$	$\left(\sqrt{6}/6, \infty\right)$
Test Value	-1	0	1
Result	-40	8	-40
Sign	-	+	-
Interpret	concave down	concave up	concave down