Exam Review: Curve Sketch, Related rates and Optimization Problems

A. Curve Sketch - $y = x^5 - 5x^4$

a) Determine the x-intercepts

$y = x^5 - 5x^4 \Rightarrow 0 = x^4(x-5) \Rightarrow x = 0$ and $x = 5$

b) Determine the y-intercepts

$y = x^5 - 5x^4 \Rightarrow y = (0)^5 - 5(0)^4 \Rightarrow y = 0$

c) Take the first derivative

$y = x^5 - 5x^4 \Rightarrow y' = 5x^4 - 20x^3 \Rightarrow 0 = 5x^3(x-4) \Rightarrow x = 0$ and $x = 4$

a. Determine coordinates of maximum and minimum points

$y = x^5 - 5x^4$

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

$x = 4 \Rightarrow y = (4)^5 - 5(4)^4 \Rightarrow y = -256 \Rightarrow (4,-256)$

b. Determine intervals where graph is increasing and decreasing

$(-\infty,0)$	$(0,4)$	$(4,\infty)$
-1	1	5
25	-15	625
$+$	$-$	$+$
$\uparrow$	$\downarrow$	$\uparrow$

d) Take the second derivative

$y' = 5x^4 - 20x^3 \Rightarrow y'' = 20x^3 - 60x^2 \Rightarrow y'' = 20x^2(x-3) \Rightarrow x = 0$ and $x = 3$

a. Determine the coordinates of the points of inflection

$y = x^5 - 5x^4$

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

$x = 3 \Rightarrow y = (3)^5 - 5(3)^4 \Rightarrow y = -162 \Rightarrow (4,-162)$

b. Determine intervals where graph is concave up or concave down

$(-\infty,0)$	$(0,3)$	$(3,\infty)$
-1	1	4
-80	-40	20
$-$	$-$	$+$
$\cap$	$\cap$	$\cup$

e) sketch the graph

B. Related rates

1. Water is being poured into an inverted cone (has the point at the bottom) at the rate of 4 cubic centimeters per second. The cone has a maximum radius of 6cm and a height of 30 cm. At what rate is the height increasing when the height is 3cm? $\left(V = \frac{1}{3}\pi r^2 h\right)$

$$\frac{r}{h} = \frac{6}{30} \Rightarrow r = \frac{6h}{30} = \frac{h}{5}$$

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h \Rightarrow V = \frac{\pi}{75}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{75}\cdot 3h^2 \cdot \frac{dh}{dt} \Rightarrow 4cm^3 = \frac{\pi}{75}\cdot 3(3cm)^2 \cdot \frac{dh}{dt}$$

2. The radius of a sphere is increasing at a rate of 2 meters per second. At what rate is the volume increasing when the radius is equal to 4 meters? $\left(V = \frac{4}{3}\pi r^3\right)$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3(4m)^2 \cdot 2m/\sec$$

3. A 20 m ladder leans against a wall. The top slides down at a rate of 4 ms^{-1}. How fast is the bottom of the ladder moving when it is 16 m from the wall?

$$c^2 = a^2 + b^2 \Rightarrow 20^2 = a^2 + 16^2 \Rightarrow 144 = a^2 \Rightarrow a = 12$$

$$c = 20m, a = 12m, b = 16m, \frac{dc}{dt} = 0, \frac{da}{dt} = -4m/\sec, \frac{db}{dt} = ?$$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} \Rightarrow 20m \cdot 0 = 12m \cdot -4m/\sec + 16m \cdot \frac{db}{dt}$$

4. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4:00 P.M.?

$$c^2 = a^2 + b^2 \Rightarrow c^2 = 240km^2 + 100km^2 \Rightarrow c^2 = 67600km^2 \Rightarrow c = 26km$$

$$c = 26km, a = 240km, b = 100km, \frac{dc}{dt} = ?, \frac{da}{dt} = 60km/hr, \frac{db}{dt} = 0$$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} \Rightarrow 26km \cdot \frac{dc}{dt} = 240km \cdot 60km/hr + 100km \cdot 0$$

5. A stone thrown into a pond produces a circular ripple which expands from the point of impact. If the radius of the ripple increases at the rate of 1.5 ft/sec, how fast is the area growing when the radius is 8 ft?

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt} \Rightarrow \frac{dA}{dt} = 2\pi(8ft)\cdot 1.5ft/\sec$$

6. If $z^2 = x^2 + y^2, \frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 3$, find $\frac{dz}{dt}$ when $x = 5$ and $y = 12$.

C. Optimization Problems

1. A shepherd wishes to build a rectangular fenced area against the side of a barn. He has 360 feet of fencing material, and only needs to use it on three sides of the enclosure, since the wall of the barn will provide the last side.

What dimensions should the shepherd choose to maximize the area of the enclosure?

2. A box with a square base has no top. If 64 cm^2 of material is used, what is the maximum possible volume for the box?